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Best approximation of periodic functions in the Lebesgue spaces

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Abstract. We obtain the exact order estimates of the best approximations of periodic functions that are analogues of the Bernoulli kernels in the Lebesgue spaces.

Keywords: *Fourier series, Bernoulli kernel, the best approximations, (ψ,β)-derivative, (ψ,β)-differentiable functions.*

Introduction. Let L_q be the space of 2π −periodic functions f summable to a power $q, 1 \le q < \infty$ (resp., essentially bounded for $q=\infty$), on the segment $[-\pi,\pi]$. The norm in this space is defined as follows:

$$\| f \|_{L_q} = \| f \|_q == \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(x)|^q dx\right)^{\frac{1}{q}}, 1 \le q < \infty, \\ \operatorname*{esssup}_{x\in[-\pi,\pi]} |f(x)|, \qquad q=\infty. \end{cases}$$

For a function $f \in L_1$, we consider its Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx},$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

are the Fourier coefficients of the function f. In what follows, we always assume that the function
$f \in L_1$ satisfies the condition

$$\int_{-\pi}^{\pi} f(x) dx = 0.$$

Further, let $\psi \ne 0$, be an arbitrary function of natural argument and let β be an arbitrary fixed real number. If a series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{i\frac{\pi}{2}\beta \operatorname{sign} k}}{\psi(|k|)} \hat{f}(k) e^{ikx}$$

is the Fourier series of a summable function, then, following Stepanets [1, p. 25] we can introduce the (ψ,β)–derivative of the function f and denote it by f_β^ψ. By L_β^ψ we denote the set of functions f satisfying this condition. In what follows we assume that the function f belongs to the class $L_{\beta,p}^\psi$ if $f \in L_{\beta,p}^\psi$ and

$$f_\beta^\psi \in U_p = \{\varphi: \varphi \in L_p, \| \varphi \|_p \le 1\},$$
$$1 \le p \le \infty.$$

If

$$\psi(|k|) = |k|^{-r}, \ r > 0, \ k \in \mathbb{Z}\setminus\{0\},$$

then the (ψ,β)–derivative of the function f coincides with its (r,β)–derivative (denoted by f_β^r) in the Weyl–Nagy sense.

Let, for a fixed function of a natural argument ψ and a number $\beta \in \mathbb{R}$, the series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx}$$

is a Fourier series of some sum function $F_\psi(x,\beta)$ on $[-\pi,\pi]$. Then each function $f \in L_{\beta,p}^\psi$ can be represented as a convolution

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x-t) F_\psi(t,\beta) dt,$$

where $\| \varphi \|_p \le 1, 1 \le p \le \infty$ (see, e.g., [2]).

Note that the functions $F_\psi(x,\beta)$ are naturally called analogs of the Bernoulli kernel, since for $\psi(|k|) = |k|^{-r}$, $r > 0$ the function $F_\psi(x,\beta)$ is the Bernoulli kernel.

By B we denote the set of functions ψ, satisfying the following conditions:

1) ψ — are positive and nonincreasing;
2) there exists a constant $C > 0$ such that

$$\frac{\psi(\tau)}{\psi(2\tau)} \le C, \tau \in \mathbb{N}.$$

Thus, the functions $\frac{1}{\tau^r}, r > 0; \frac{\ln^\gamma(\tau+1)}{\tau^r}, \gamma \in \mathbb{R}$, $r > 0, \tau \in \mathbb{N}$, and some other functions belong to the set B.

For the quantities A and B, the notation $A \asymp B$ means that there exist positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. If $B \le C_2 A (B \ge\ge C_1 A)$, than we can write $B \ll A (B \gg A)$. All $C_i, \ i = 1,2,\ldots$, encountered in our paper may depend only on the parameters appearing in the definitions of the class and metric in which we determine the error of approximation.

We now give definitions of the approximating characteristic under investigation.

Let

$$T_m = \left\{t: t(x) \sum_{k=-m}^{m} c_k e^{ikx}\right\}.$$

For $f \in L_q, 1 \le q \le \infty$, we set

$$E_m(f)_q = \inf_{t\in T_m} \| f(\cdot) - t(\cdot) \|_q. \qquad (1)$$

The quantity given by relation (1) is called the best approximation of the function f in the space L_q.

In the case where $c_k = \hat{f}(k)$, by $\mathcal{E}_m(F_\psi)_q$ we denote the quantities

$$\mathcal{E}_m(f)_q = \| f(\cdot) - \sum_{k=-m}^{m} \hat{f}(k) e^{ikx} \|_q.$$

For the quantities $E_m(f)_q$ and $\mathcal{E}_m(f)_q$ there is the relations

$$E_m(f)_q \le \mathcal{E}_m(f)_q.$$

At present, there are many works devoted to the investigation of quantity $E_m(f)_q$ and $\mathcal{E}_m(f)_q$. For details and the corresponding references, see, e.g., [3, 4].

Main result. The following assertion is true:

Theorem. Let $\mathbf{1 < q < \infty}$, $\boldsymbol{\psi \in B}$, $\boldsymbol{\beta} \in \mathbb{R}$ and let, in addition, there exist $\boldsymbol{\varepsilon > 0}$ such that the sequence $\boldsymbol{\psi(t) t^{1-\frac{1}{q}+\varepsilon}}$, $\boldsymbol{t} \in \mathbb{N}$, does not increase. Then the following order estimate is true:

$$\mathcal{E}_m(F_\psi)_q \asymp E_m(F_\psi)_q \asymp \psi(m) m^{1-\frac{1}{q}}.$$

Proof. We now establish the upper bound. Let l and m be such that $2^l < m \le 2^{l+1}$. First, we consider the case $1 < q \le 2$. Applying Littlewood–Paley Theorem (see, e.g., [5])

$$C_3(q) \parallel f \parallel_q \le \left\| \left(\sum_s |\delta_s(f,\cdot)|^2 \right)^{\frac{1}{2}} \right\|_q \le$$
$$\le C_4(q) \parallel f \parallel_q,$$

where

$$\delta_s(f,x) = \sum_{k \in \rho(s)} \hat{f}(k) e^{ikx},$$
$$\rho(s) = \{k: 2^{s-1} \le |k| < 2^s\},\ s \in \mathbb{N},$$

we obtain

$$E_m(F_\psi)_q \ll \left\| F_\psi - \sum_{s<l} \delta_s(F_\psi) \right\|_q =$$
$$= \left\| \sum_{s \ge l} \delta_s(F_\psi) \right\|_q \ll$$
$$\ll \left\| \left(\sum_{s \ge l} |\delta_s(F_\psi)|^2 \right)^{\frac{1}{2}} \right\|_q = I_1.$$

Then, using the inequality $|a+b|^\alpha \le |a|^\alpha + |b|^\alpha$ for $0 \le \alpha \le 1$, we can write

$$I_1^q \ll \int_\pi^\pi \sum_{s \ge l} |\delta_s(F_\psi)|^q dx \ll$$
$$\ll \sum_{s \ge l} \parallel \delta_s(F_\psi) \parallel_q^q,$$

and.

$$I_1 \ll \left(\sum_{s \ge l} \parallel \delta_s(F_\psi) \parallel_q^q \right)^{\frac{1}{q}}.$$

To continue (2), estimate the value

$$\parallel \delta_s(F_\psi) \parallel_q = \left\| \sum_{k \in \rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \right\|_q.$$

First, we show that the following estimate is true

$$\left\| \sum_{k \in \rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \right\|_q \ll$$
$$\ll \psi(2^s) \left\| \sum_{k \in \rho(s)} e^{ikx} \right\|_q, 1 < q < \infty.$$

For this purpose, for $s \ge l$ we consider the sequence

$$\{\lambda_k\} = \left\{ \frac{\psi(|k|)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k},\ 2^{s-1} \le |k| < 2^s \right\}.$$

We make sure that the sequence $\{\lambda_k\}$ satisfies the conditions of the Marcinkiewicz theorem (see, e.g., [5]).

Obviously, it is enough to check the fulfillment of conditions 1), 2) of this theorem for positive k such that $2^{s-1} \le k < 2^s$.

By $\psi \in B$ and $2^{s-1} \le k < 2^s$ then

$$1) |\lambda_k| = \left| \frac{\psi(k)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta} \right| = \frac{\psi(k)}{\psi(2^s)} \le$$
$$\le \frac{\psi(2^{s-1})}{\psi(2^s)} \le M,$$

$$2) \sum_{k=2^{s-1}}^{2^s-1} |\lambda_k - \lambda_{k+1}| =$$

$$= \sum_{k=2^{s-1}}^{2^s-1} \left| \frac{\psi(k)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta} - \frac{\psi(k+1)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta} \right| \le$$

$$\le \frac{1}{\psi(2^s)} \sum_{k=2^{s-1}}^{2^s-1} (\psi(k) - \psi(k+1)) =$$
$$= \frac{1}{\psi(2^s)} (\psi(2^{s-1}) - \psi(2^s)) \le$$

$$\le \frac{\psi(2^{s-1})}{\psi(2^s)} \le M.$$

Therefore,

$$\Lambda_s \sum_{k \in \rho(s)} e^{ikx} =$$
$$= \sum_{k \in \rho(s)} \frac{\psi(|k|)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} =$$
$$= \frac{1}{\psi(2^s)} \sum_{k \in \rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx}.$$

So we can write

$$\left\| \Lambda_s \sum_{k \in \rho(s)} e^{ikx} \right\|_q =$$
$$= \frac{1}{\psi(2^s)} \left\| \sum_{k \in \rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \right\|_q.$$

On the other hand, by Littlewood–Paley Theorem, there is an estimate

$$\left\| \Lambda_s \sum_{k \in \rho(s)} e^{ikx} \right\|_q \le C_5(q) M \left\| \sum_{k \in \rho(s)} e^{ikx} \right\|_q.$$

So, in the end we get

$$\parallel \delta_s(F_\psi) \parallel_q =$$
$$= \left\| \sum_{k \in \rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \right\|_q \ll$$
$$\ll \psi(2^s) \left\| \sum_{k \in \rho(s)} e^{ikx} \right\|_q.$$

Further, by using the last estimate and the well-known relation (see, e.g., [4])

$$\left\| \sum_{k \in \rho(s)} e^{ikx} \right\|_q \asymp 2^{s\left(1-\frac{1}{q}\right)}, 1 < q < \infty,$$

we get

$$\parallel \delta_s(F_\psi) \parallel_q \ll \psi(2^s) 2^{s\left(1-\frac{1}{q}\right)}. \qquad (3)$$

Combining relations (2) and (3), we can write

$$I_1 = \left\| \sum_{s \ge l} \delta_s(F_\psi) \right\|_q \ll \left(\sum_{s \ge l} \psi^q(2^s) 2^{qs\left(1-\frac{1}{q}\right)} \right)^{\frac{1}{q}}.$$

By the condition of the theorem, there exists $\varepsilon > 0$ such that the sequence $\psi(t)t^{1-\frac{1}{q}+\varepsilon}$ does not increase. Hence, we can write

$$E_m(F_\psi)_q \ll I_1 \ll$$

$$\ll \psi(2^l) 2^{l\left(1-\frac{1}{q}+\varepsilon\right)} \left(\sum_{s \ge l} 2^{-s\varepsilon q} \right)^{\frac{1}{q}} \ll$$

$$\ll \psi(2^l) 2^{l\left(1-\frac{1}{q}\right)} \asymp \psi(m) m^{1-\frac{1}{q}}.$$

Now consider the case $2 < q < \infty$. Applying Littlewood–Paley Theorem and the Minkowski inequality, we obtain

$$\left\| F_\psi - \sum_{s<l} \delta_s(F_\psi) \right\|_q = \left\| \sum_{s \ge l} \delta_s(F_\psi) \right\|_q \ll$$

$$\ll \left\| \left(\sum_{s \ge l} |\delta_s(F_\psi)|^2 \right)^{\frac{1}{2}} \right\|_q =$$

$$= \left(\left\| \sum_{s \ge l} |\delta_s(F_\psi)|^2 \right\|_{\frac{q}{2}} \right)^{\frac{1}{2}} \ll$$

$$\ll \left(\sum_{s \ge l} \left\| |\delta_s(F_\psi)|^2 \right\|_{\frac{q}{2}} \right)^{\frac{1}{2}} =$$

$$= \left(\sum_{s \ge l} \| \delta_s(F_\psi) \|_q^2 \right)^{\frac{1}{2}}.$$

Next, using (3) and repeating the considerations made for the case $1 < q \le 2$, we obtain the required estimate

$$E_m(F_\psi)_q \ll \psi(m) m^{1-\frac{1}{q}}, \qquad 2 < q < \infty.$$

Thus, the required upper bound is established.

We now determine the lower bounds. Let $t^* \in T_m$ be the polynomial of the best approximation of the function $F_\psi(x, \beta)$ in the space $L_q, 1 < q < \infty$, that is

$$E_m(F_\psi)_q = \inf_{t \in T_m} \| F_\psi - t \|_q = \| F_\psi - t^* \|_q,$$

and

$$F_2(x, \beta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-2} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx}.$$

Consider the quantity

$$J = (F_\psi - t^*, F_2 - S_m(F_2)) = (F_\psi, F_2 - S_m(F_2)) -$$
$$-(t^*, F_2 - S_m(F_2)) = (F_\psi, F_2 - S_m(F_2)).$$

On the one hand, by Hölder's inequality, we can write

$$J \le \| F_\psi - t^* \|_q \| F_2 - S_m(F_2) \|_{q'} =$$
$$= E_m(F_\psi)_q \| F_2 - S_m(F_2) \|_{q'},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Since (see, e.g., [3]),

$$\| F_2 - S_m(F_2) \|_{q'} \ll 2^{-m\left(2-\frac{1}{q}\right)},$$

then

$$J \ll E_m(F_\psi)_q 2^{-m\left(2-\frac{1}{q}\right)}. \qquad (4)$$

On the other hand, for the value of J we can write

$$J \gg \sum_{k \in \mathbb{Z} \setminus \{0\}} \psi(|k|)|k|^{-2} \gg$$

$$\gg \sum_{s \ge m} \sum_{k \in \rho^+(s)} \psi(k) k^{-2} =$$

$$= \sum_{s \ge m} \sum_{k=2^{s-1}}^{2^s} \psi(k) k^{-2} \gg$$

$$\gg \sum_{s \ge m} \psi(2^s) 2^{-s} \gg \psi(m) 2^{-m}, \qquad (5)$$

where $\rho^+(l) = \{k : 2^{l-1} \le k < 2^l\}$.

Taking into account relations (4) and (5), we obtain

$$\psi(m) 2^{-m} \ll J \ll E_m(F_\psi)_q 2^{-m\left(2-\frac{1}{q}\right)},$$

thus

$$E_m(F_\psi)_q \gg \psi(m) m^{1-\frac{1}{q}}.$$

The required lower bound is established, which proves the theorem.

Conclusions. We establish the exact order estimates of the best approximations of periodic functions that are analogues of the Bernoulli kernels in the space $L_q, 1 < q < \infty$.

The assertion of Theorem for $\psi(|k|) = |k|^{-r}$, $r > 1 - \frac{1}{q}, 1 \le q \le \infty$ were established by Temlyakov V. N. [3]

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