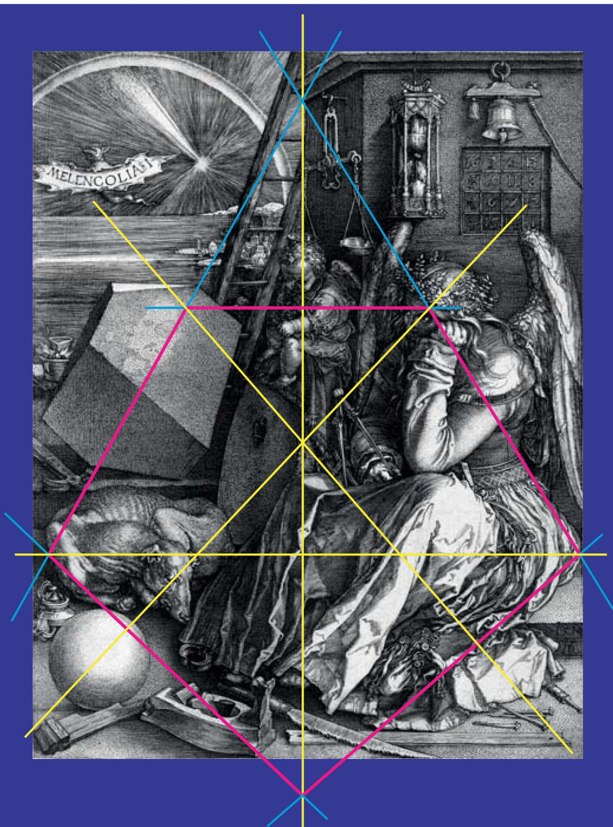
CURRENT STATUS AND PROSPECTS OF MATHEMATICAL EDUCATION

MONOGRAPH



BUDAPEST

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This book will be of interest to all researchers in the field of didactics of mathematics.

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CONTENTS

PREFACE

Reforming and modernizing processes in the educational system bring about the necessity of the search for ways and means of its reorientation towards holistic development, upbringing and socialization of an individual capable of living in a society, being able to support civilized interaction with nature, having a desire for self-improvement and life-long learning, and being ready for a conscious life choice and self-actualization, responsibility, work activity, and civic activity.

It is publicly recognized that the quality of the mathematical training is a decisive factor in the overall development of the individual, as well as an indicator of the society's readiness for sustainable socio-economic growth and the introduction of high technologies in all spheres of human life and activity.

At the same time, the crisis phenomena continue to deepen in the system of mathematical education at its all levels,. Among other things, we witness the fall in the prestige of mathematical education and mathematics-related professions; formalism in the knowledge of University and high school students; their inability to apply the mathematical apparatus to solving real life problems; insufficient computing culture of students and pupils.

The issues of the organic combination of fundamental and professional components in the content of mathematical training in tertiary and secondary education, ensuring the continuity of the content and coordination of the learning process at the levels of basic secondary, specialized, vocational, and higher education, improvement of the organization of educational activity of University and high school students, search for and introduction of effective teaching technologies, particularly computer-based ones, the development of new electronic teaching aids, introducing practical and applied tasks into mathematical courses at various levels of mathematical education are the topical ones.

Adoption of the competence approach as a methodological basis of the educational process shifts priorities in mathematical training of students and schoolchildren, the social focus is moved from the knowledge, abilities, and skills acquisition to their transformation into competencies and competences via filling them with value attitudes as well as socially and personally significant experience of successful application.

Thus, the time has come for a new scientific rethinking of all the components of the methodological system of mathematical education of youth at different levels.

The monograph is an exposure to some aspects of this multifaceted problem.

CHAPTER 1. SOME CONCEPTUAL ISSUES

1.1. Functioning and Development of Different Systems of Arranging Students' Educational-Cognitive Activity in Learning Mathematics

Larysa Golodiuk

Introduction

Teacher and students are the participants of the educational-cognitive activity. In our opinion, the interaction between them may be built within such systems:

- «a teacher as a direct subject – students as a collective subject – a mathematical object of cognition»;
- «a student / student as individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject»;
- «a student as an individual subject – groups of students as collective subjects – a mathematical object of cognition – a teacher as direct and indirect subject»;
- «a student as an individual subject – a mathematical object of cognition – a teacher as an indirect subject».

First of all, we are to note, that the system is a set of elements that stay in certain relationships and relations to each other and form a certain integrity, unity[1] (Philosophic dictionary, 2006). In the above mentioned systems the relations are arranged among various elements (a student / students as a collective subject / a group of students; a teacher as a direct subject / a teacher as an indirect subject; a mathematical object of cognition, materialized presentation of information about mathematical content / mathematical object of cognition of the materialized presentation of information about mathematical content / mathematical object of cognition of symbolic presentation of information about mathematical content / mathematical object of cognition of mixed forms of presentation of information about mathematical content), which in its turn is a manifestation of these systems in common and distinguishing features.

[1] *Philosophic dictionary* (2006). Kiev, Ukraine: ASK. (in Ukr.).

To the common features we refer the following:

- the subordination of the functioning of the above mentioned systems of the process of arranging the students' educational-cognitive activity;
- the development of each system towards gaining integrity as intrinsic property that, being in a stage of relative stability, emphasizes the perfection and the completeness of the initial, stage of the development of each system;
- functioning and development of the systems are ensured by building internal connections on the basis of subject-subject and subject-object interaction between the system elements.

Thereby, the organization of students' educational-cognitive activity provides the functioning and development of each of the systems «a teacher as a direct subject – students as a collective subject – a mathematical object of cognition»; «a student / student as individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject»; «a student as an individual subject – groups of students as collective objects – a mathematical object of cognition – a teacher as a direct and indirect subject»; «a student as an individual subject – a mathematical object of cognition – a teacher as an indirect subject». In this case the operation is aimed at maintaining the subject-subject and subject-object interaction between the system elements; and the development is aimed both – at reaching the moment when processes, that characterize the ascending and descending phases of the system, come in a state of relative equilibrium, and at the transition of the system to a new level of functioning.

Analysis of recent research and publications showed that the problem of students' educational-cognitive activity is a subject of scientific research of national scientists. Concerning the arrangement of educational-cognitive activity in basic school, the significant research has been made by N. Tarasenkova[2] (1991) (stimulating students' cognitive activity at school math lectures and practical

[2] Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Unpublished Doctoral dissertation). Kiev state pedagogical university, Kiev, Ukraine. (In Rus.).

training); T. Arkhipova[3] (2002) (monitoring the process of enhancing students' learning and cognitive activities in 7th–9th grades in the computer-based geometry); L. Kormina[4] (1998) (generating philosophical knowledge in the process of educational-cognitive activity of the adolescents); V. Tatochenko[5] (1999) (formation techniques of mental activity in learning mathematics (6th–8th grades)); O. Tregubenko[6] (2005) (defining didactic prerequisites for the ideas about the objects and phenomena of reality in a secondary school extracurricular learning activities (5th–6th grades)).

However, without denying the significant contribution to the solution of this problem made by the above-mentioned authors, it should be noted that the process of arranging students' educational-cognitive activity in math class requires methodological and technological refinements. Thus, the aim of the publication is to reveal the functioning and development of systems of interaction between the subjects of study and define the constituent elements of the math teachers' readiness to organize students' educational-cognitive activity in the 5th–9th grades.

Presentation of the basic material

Disclosure of the functioning and development of each system will be made by focusing on the subject-subject and subject-object interaction between the system elements.

To implement the process of the organization of students' educational-cognitive activity in the context of learning mathematics in curricular and extracurricular time *in*

[3] Arkhipova, T. L. (2002). *Stimulation of students' cognitive activity in the process of Geometry teaching using the computer* (Unpublished Doctoral dissertation). Kiev state pedagogical university, Kiev, Ukraine. (in Ukr.).

[4] Kormina, L. I. (1998). *Formation of ideological knowledge of adolescents in the process of educational and cognitive activity* (Unpublished Doctoral dissertation). Lesya Ukrainka Eastern European National University, Lutsk, Ukraine. (in Ukr.).

[5] Tatochenko, V. I. (1999). *The method of forming the intellectual activity in the teaching of mathematics to the 6th-8th grades students* (Unpublished Doctoral dissertation). Kiev state pedagogical university, Kiev, Ukraine. (in Rus.).

[6] Tregubenko O. M. (2005). *Formation of the 5th-6th grades pupils' perceptions about the objects and phenomena of the surrounding reality in the educational and cognitive activity of the secondary school* (Unpublished Doctoral dissertation). Lugansk National Taras Shevchenko University, Lugansk, Ukraine. (in Ukr.).

the system «a teacher as a direct subject – students as a collective subject – the mathematical object of cognition», teacher's readiness to implement the system of techniques and operations of communication with the students, which ensures a communicative process based on the discussion, during which the active position of its participants is reinforced, is of particular importance; it provides the setting of the objectives, the allocation of the responsibility for achieving them (if necessary), it establishes the optimal emotional climate in the process of communication, discloses personal potential of its participants and the successful bringing into effect of the educational-cognitive activity.

The effectiveness of functioning of the analyzed system, taking into account the subject-subject interaction between the system elements determined the following: conceptual-cognitive resource of the teachers who, on the one hand, represent knowledge about communication providing the interaction of the educational-cognitive activity participants, is aimed at the harmonization and unification of their efforts to establish relations and reach a common result, and on the other hand, there are the analysis and evaluation of relevant methods and techniques of communication, the prediction of the consequences of their application, the implementation of the current monitoring, assessment and adjustment of their communicative behavior; presence of the productive communication setup, which testifies to the practical significance of the activities offered to students; active participation of the participants of educational-cognitive activity in the process of the establishing of the objectives and aims of activities, or students' awareness of the goal formulated by the teacher; presence of the achievement motivation which can be worded this way: «I want to teach myself…» (Miier[7], 2016).

The development of the system «a teacher as a direct subject – students as a collective subject – a mathematical object of cognition» is based on the fact that a collective subject is a subject which is practically realized through the

[7] Miier T. I. (2016) *Organization of educational and research activity of junior pupils. Monograph.* Kirovograd, Ukraine: FO-P Aleksandrova M. V. (in Ukr.).

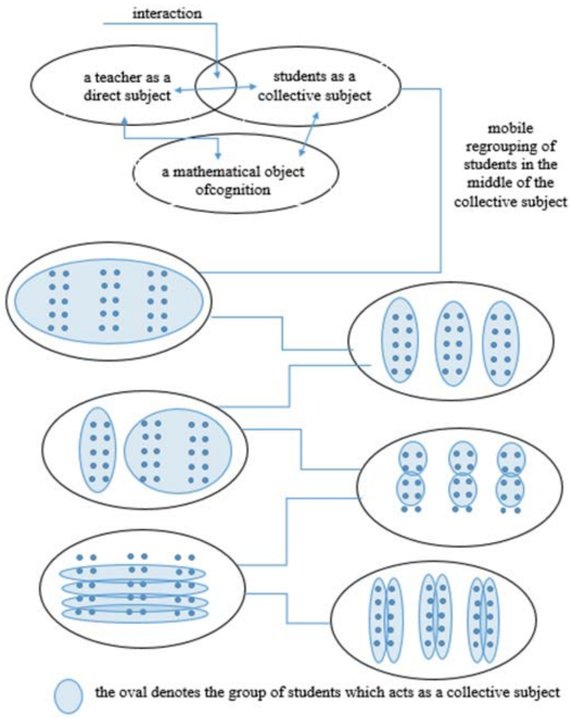

Figure 1.1.1 Organization of educational-cognitive activity of students in the system «a teacher as a direct subject – students as a collective subject, a mathematical object of cognition» on the basis of mobile regrouping of students in the middle of the collective subject.

efforts of many individual subjects, but it is not limited to them and is relatively autonomous from them.

Given the above, the collective subject should be considered by the teacher as a certain amount of students, who are not dissolved in each other, but are flexibly interconnected for the organization of learning and research activities. Mobility shows itself up in the optimum regrouping of students in the middle of the collective subject, which is actualized with regard to their mental abilities (cognitive styles) and in the framework of the collective subject that cognizes mathematical object of material, reified, symbolic, and mixed forms of the representation of mathematical content (Golodiuk[8], 2017).

[8] Golodiuk, L. S. (2017). *The organization of educational and cognitive activity of secondary school pupils in learning mathematics at the lessons and in extracurricular time: theoretical aspect Monograph.* Kropivnytsky, Ukraine: FOP Aleksandrova M. V. (in Ukr.).

The mathematical object of cognition can be represented by an object of material, reified, symbolic, and mixed forms of presentation of mathematical content. Conditionally, it is shown in Figure 1.1.2.

In Figure 1.1.2 the arrows show the possible transition from one form of presenting mathematical information to another. For example, primarily the students study the subject of the environment through the use of visual analysis, then they find out the materialized presentation of mathematical object (using the picture), then from the set of symbolic means they clarify the nformation, namely: they indicate the main elements, and then move to the study of the mixed forms of the presentation of mathematical information.

The functioning and development of *the system «a student / student as individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject»* in the process of organization of students' educational-cognitive activity allows for their dialogue, the content of which is determined by the process of learning a mathematical object. Students' activity is organised on the basis of an exchange of individual interrelated messages. This process is accompanied by critical evaluation of their own utterances and a critical attitude towards statements made by another person; it requires a considerable concentration, which, in its turn,

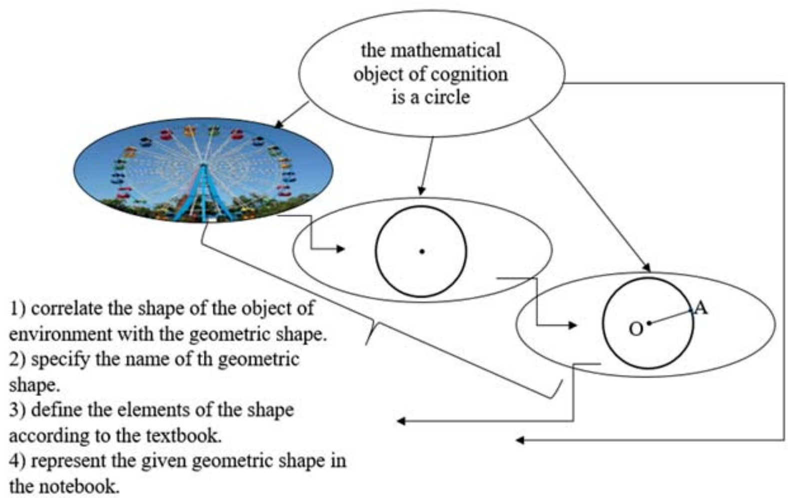

Figure 1.1.2 The transition of one form of representation of mathematical content into another.

provides the focus on the object of cognition, as well as on a partner of activities, which provide for joint implementation of actions, and the obtained results.

An important component of functioning and development of the system of «a student / student as individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject» is students' alternate implementation of different roles, including roles, «I am a Communicator» and «I am a Recipient» (Figure 1.1.3).

After all, the holistic dialogue is possible when listening to the interlocutor and facilitating the continuation of his thoughts is provided. Specified becomes possible if each participant of a future dialogue mentally pronounce the following: «During a conversation, I have the opportunity to show myself as an attentive, tolerant, knowledgeable, motivated person and work hard to ensure that...» (there are possible options to continue sentences more consistently to express their thoughts, to speak more convincingly; not to distract during a conversation; to listen actively, and therefore to ask the interlocutor questions on the content of his/her remarks, etc.).

The dialogue, which is preceded by a pre-specified mental preparation, is more likely to line up as constructive, that is, one that promotes intellectual growth of its partners, even though each of them is learning to listen to the partner, to hear the opinions of others, to evaluate and to rethink one's own thoughts, to compare and analyze them with the interlocutor's thoughts. The creation of a constructive dialogue is supported by students' performing different exercises, among which the perceptual (they learn to create the dialogue), imitation (they repeated samples of dialogical speech), analytic-synthetic (they analyzed the topic of the future dialogue, predicted replicas that will contribute to both its implementation and will support a productive dialogue; learned to ask questions, formulate answers, analyzed the content of the questions and formulated answers), exercises on the observance of speech etiquette.

Figure 1.1.3 is a conventional image of the functioning and development of the system of «a student /student as individual-collective subject – a mathematical object of

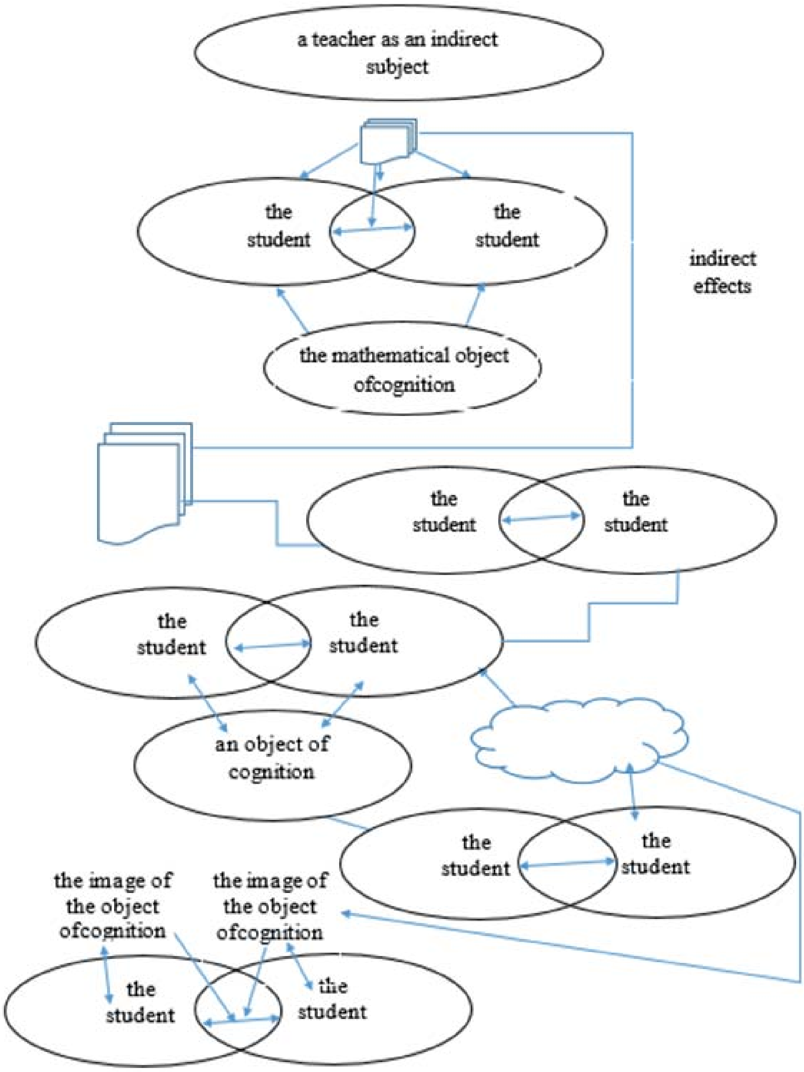

Figure 1.1.3 The functioning and development of the system « a student / student individual-collective subject – object of cognition – a teacher as an indirect object» in the process of organization of students' educational-cognitive activity.

cognition – a teacher as an indirect object», which is based on teacher's indirect influence on students' educational activities. An important component of functioning and development of this system is the indirect managing students' activities, which includes steps for structuring information about a mathematical object of cognition in a visual way.

In our interpretation, the visual image of the object of cognition is a side result of the implementation of students' educational-cognitive activity and is a series of images that are consecutively arranged, adjusted and supplemented by students with new information in a condition of reflexive analysis of the educational situation in order to reflect a generalized reflection of their own vision of the essence of the object of cognition in pictures, based on the performance of several cognitive acts. Visual images of the objects of cognition testify to the adequacy and completeness of the image of concepts, which, being direct products of educational-cognitive activity, reflect the nature of objects of cognition and are the basis for the mastery of mathematical concepts and the formation of educational-cognitive skills.

In the context of students' educational-research activity structuring serves the purpose of organizing the scattered information in the process of memorising, in which the elements of the material that is being memorized, are associated in meaning in the whole group or several groups of the kind. Methods of structurization of the educational information, used by the students in the system «a student – student – a mathematical object of cognition – a teacher as an indirect subject» are actions with the information represented by the terms «information coagulating», «information coding».

The functioning and development of the system «a student – student – a mathematical object of cognition – a teacher as an indirect subject» is provided by students' active work on the results of scientists' operations of information coagulating. This is presented in the textbook rules, theorems, which contain the most important and most substantial knowledge about the objects of cognition. In addition, students are engaged in the unassisted activities concerning the information coagulating. The implementation of these actions is not aimed at a simple reduction of the amount of information about the mathematical object of cognition, but at the eliminating of the irrelevant information. The process of coagulation of information is carried out by the students in two stages:

1) separation of the secondary information from the essential and basic information;

2) summarizing the extracted information, as such, that contains enough information for its further deployment, that is, preserving the informative value of the message.

Also in the system «a student / student – a mathematical object of cognition – the teacher as an indirect subject» students carry out action-coding information about the mathematical object of cognition making use of signs and symbols, drawings (reproduction of information about a mathematical object of cognition in the form of a drawing or a consistent set of drawings) and color coding (highlighting essential information; locating similar content material against the background of the same color; the contrast allocation of the opposite information, which is one unit of information about a mathematical object of cognition).

The teacher as an indirect subject of the analyzed system of organization of educational-research activity provides students with the necessary regulations, instructions, advice, which direct students' activities and facilitate independent getting the results in the shortest time period of the implementation of the educational-cognitive activity.

The functioning and development of *the system «a student as an individual subject – groups of students as collective subjects – a mathematical object of cognition – a teacher as a directly-indirect subject»* in the process of organization of educational-cognitive activity of students allows for the accounting of the actual development of mathematical and mental abilities of a student, some students, groups of students. Such student (some students or group of students where each student considered separately, i.e. as individual subject) acquires (students acquire) a rank «the students as the individual subject», which in the process of implementation of activities works independently and is developing (before the students start working on this task) on order of the implementation of cognition of the mathematical object on the basis of cognitive acts, which later serves as a basis for the work of groups of students as collective subjects (Figure 1.1.4).

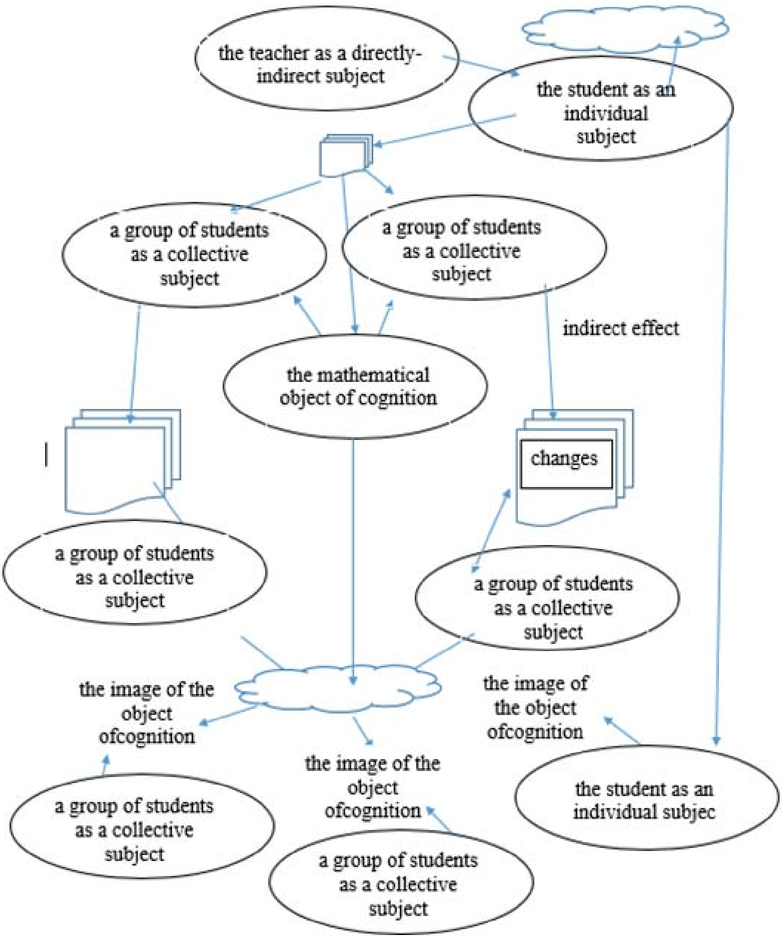

Figure 1.1.4 The functioning and development of the system «a student as an individual subject – groups of students as collective subjects – a mathematical object of cognition a teacher as a directly-indirect subject» in the process of organization of educational-cognitive activity of students.

Groups of students as collective subjects are formed during unassisted and conscious election by the students of one of the ways of discovering one's own cognitive activity within the framework of the collective subject. In the system «a student as an individual subject – groups of students as collective subjects – a mathematical object of cognition – a teacher as a directly-indirect subject»there are effective ways of manifestation of cognitive activity: reproductive (performing actions as directed in the implementation of cognition of the mathematical object on the basis of cognitive acts which encouraged student – an individual subject) and creative (changes in regulations for the implementation of the cognition of mathematical object, based on the cognitive acts developed by the individual subject, and the rationale for the changes that are being proposed to regulations.

The teacher as a directly-indirect subject, if necessary, carries out a direct management of the process of «making» by the student as an individual subject to the requirements of the implementation of cognition of the mathematical object on the basis of cognitive acts, because an improper regulation will obstruct the work of groups of students as collective subjects, and provides indirect management of groups of students by asking leading questions, updating necessary knowledge, which will contribute to further activities of students.

The student, as an individual subject can realize educational-cognitive activity as a particular stage of the lesson (for example, when the rest of the class are doing an independent work or homework of academic work as one of its tasks). In the vast majority of cases the student as an individual object preferred to perform learning activities in extracurricular time, i.e. doing the home assignments, but the results were presented at the next lesson.

The functioning and development of *the system «a student as an individual subject – a mathematical object of cognition – a teacher as an indirect subject»* in the process of organization of students' educational-cognitive activity was envisaged the student's independent educational-cognitive activity, with a gradual transition from the reproductive management method of manifestation of cognitive activity to creative and then to the oversituational one. The transition is performed with the indirect influence of the teacher, resulting in providing the student with the opportunity to use pedagogical support, which, in students' opinion, is optimal. This can be a piece of advice, formulating together with the teacher the instructions for the performance of the educational-cognitive activity, the selection (in partnership with the teacher) of the sensory standards of the cognition of a mathematical object, remembering the necessary information, etc.

The student as an individual subject has to practice in the independent: implementation of a sensory-perceptual display of information of mathematical content in the images of objects of cognition based on the use of sensory standards; implementation of reproductive and creative use of mental images of objects of cognition for mastering mathematical knowledge; implementation of the educational-cognitive activity based on tasks and problems; performing various ways of the educational-cognitive activity, namely: perceptual, problem-oriented, search, variation, activity-observation, heuristic, research, learning research, design, graphic, practical and modeling activities; creating a visual image of cognition of a mathematical object on the basis of images of the object of cognition reflected in the views **(Figure 1.1.5).**

Activities should be accompanied by appropriate pedagogical support, which does not stop student's cognitive activity, and is adjusted to implement the educational-cognitive activity in a necessary direction.

Thus, the process of organization of students' educational-cognitive activity in learning mathematics is built on the basis of functioning and development of such systems:

- «the teacher as a direct subject – students as a collective subject, a mathematical object of cognition»;
- «a student / student as individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject»;
- «the student as an individual subject – groups of students as collective subjects – the mathematical object of cognition is the teacher as a directly-indirect subject»;
- «the student as an individual subject, a mathematical object of cognition – a teacher as an indirect subject» that requires teacher's willingness to implement it.

Based on the fact that the willingness of mathematics teachers to organize students' educational-cognitive activity in the 5^{th}–9^{th} grades is interpreted by us as a holistic formation, a special mental condition that is characterized by the presence of the image of the educational-cognitive activity in their mind, the process of the organization to meet the

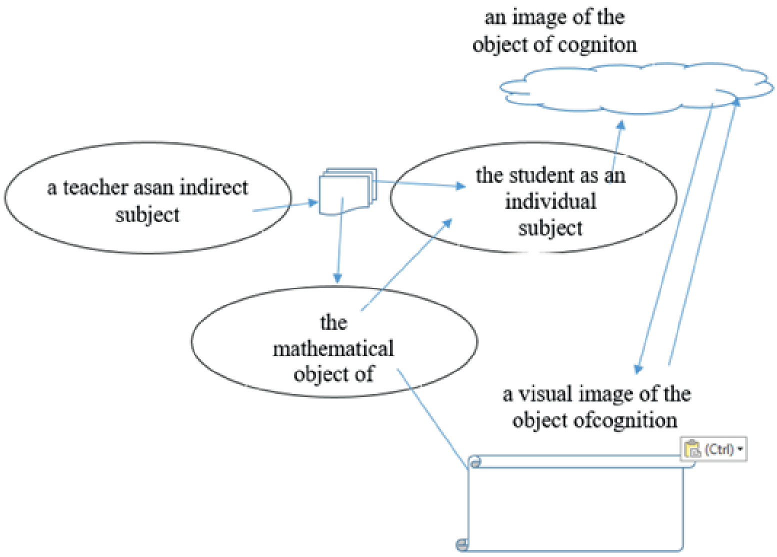

Figure 1.1.5 The functioning and development of the system «student as an individual subject – a mathematical object of cognition – a teacher as an indirect subject» in the process of organization of students' educational-cognitive activity.

stated purpose and goals; the image of the structure of the learning activities as a set of actions, which according to the objectives of the activities are categorized as cognitive acts, in which students carry out learning activity; the image of the way of activity, which are used to organize students' educational-cognitive activity. Readiness as an integral formation contains motivational, substantial and procedural components, which, taken together, provide an effective organization of educational-cognitive activity of students of the 5^{th}–9^{th} grades in mathematics class and in their free time. Further, we will consider these components of preparedness.

The motivational component of the preparedness in focus is related to the sphere of needs and motivation, which, according to S. Rubinstein[9] (1989), is the «core of personality». The above said has to do with the teacher, and therefore we concretize needs and motivations in the context of teacher's professional activity and the focus of scientific research.

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[9] Rubinstein, S. L. (1989). *Fundamentals of General Psychology*. Moscow, Russia: Pedagogika. (in Rus.).

The concept of «sphere of needs and motivation» contains a significant number of components, we shall focus on two of them. These are needs and motives. By definition given by L. Bozhovich[10] (1972), the need indirectly stimulates a person to seek satisfaction, and the process of this search is coloured with positive emotions. According to A. Leontiev[11] (1975), the problem of needs in their reality is always behind the motive, but in the concept of «need» the *motive* is hidden.

The interpretation of another component of the sphere of needs and motivations, namely: motive, is also charac-terized by various semantic contents. For example, L. Bozhovich[12] (1972) under the *motive* understands «a number of special incentives of human behavior. As for a motive, there may be objects of the external world, perceptions, ideas, feelings and experiences. That is all what embodies the need». According to A. Leontiev[13] (1975), «a motive is an object that meets that need and which, in one form or another, reflected by the subject, leads it to activity». In the interpretation of other psychologists motive appears as «a complex mental education, which is to be built by the subject themselves» (Ilyin[14], 2000); «the incentive to activities associated with the satisfaction of human needs» (Savchyn[15], 2007).

It is important to focus our attention on the formation of the motive. In this process E. Ilyin distinguishes three stages and the components which correspond to them. Wish, need, desire, drive, passion are referred to the first stage, which should explain why the person was motivated to do something. The second stage, that of the «internal filter», is the one where the sorting out of purposes and

[10] Bozhovich, L. I. (1972). *The problem of the development of the child's motivational sphere*. Moscow, Russia: Pedagogika. (in Rus.).

[11] Leontiev, A. N. (1975). *Activity. Consciousness. Personality*. Moscow, Russia: Politizdat. (in Rus.).

[12] Bozhovich, L. I. (1972). *The problem of the development of the child's motivational sphere*. Moscow, Russia: Pedagogika. (in Rus.).

[13] Leontiev, A. N. (1975). *Activity. Consciousness. Personality*. Moscow, Russia: Politizdat. (in Rus.).

[14] Ilyin, E. P. (2000). *Motivation and motives*. Saint Petersburg, Russia: Peter. (in Rus.).

[15] Savchyn, M. V. (2007). *Pedagogical psychology*. Kiev, Ukraine: Academic Edition. (in Ukr.).

ways of their achievement takes place, and is associated with the tendencies, level of claims, taking into account the knowledge, skills, ideals, beliefs, attitudes, and values. The semantic load of this block is the explanation of why this incentive was implemented exactly in this way (or why the subject refused to meet the need). The third stage, which is called the «target stage», is associated with the motivation to achieve the goal, desire, or intention. The essence of the semantic load is why a particular action or act is performed, what its meaning is (Ilyin[16], 2000).

Therefore, the motivational component of the math teacher's readiness to organize the students' educational-cognitive activity in the 5th-9th grades «is built» by the teacher themselves, it encourages them to start the process, which allows for the implementation of the students' unassisted, conscious, and effective cognition of mathematical objects of material, reified, symbolic, and mixed forms of representation based on the implementation of several cognitive acts.

The basement of this component is the cognitive motives (caused by the perceptions about the organization of the current process of training students, the level of self-training, a feeling of the need for new knowledge (innovation)) and professional motives (caused by the need to achieve success in professional activities, getting recognition among colleagues, students, and parents).

A content component of readiness of the math teacher to organize the students' educational-cognitive activity in the 5th–9th grades concerns the knowledge which lays the foundation to the organization of the process under analysis by the teacher.

The disclosure of a content component of the preparedness being analyzed is feasible, taking into account the following key ideas of scientific research:

- activity is a form of performance, which, on the one hand, is manifested in the system of actions, and on the other hand it is governed by a conscious purpose and is directed at its accomplishment (Leontiev[17], 1975);

[16] Ilyin, E. P. (2000). *Motivation and motives.* Saint Petersburg, Russia: Peter. (in Rus.).

[17] Leontiev, A. N. (1975). *Activity. Consciousness. Personality.* Moscow, Russia: Politizdat. (in Rus.).

- the phenomenon of the «organization of students' activity» is revealed in such semantic lines:

    1) organization as the process, which is carried out for a purpose;
    2) organization as the process of governance (governance is the process of interaction of subsystems that act as subject and object control);
    3) organization of the process monitoring (monitoring as a supervisory function is provided by students' activity through direct or indirect means of organization of their cognitive self-support and activity in interaction with other, additional issues that agree on the reasoning, using cards-instructions (cards, certificates), diagrams, tables, algorithms, regulations, etc) (Miier[18], 2016);

- students' educational-cognitive activity is a collection of the related educational (learning) actions, the implementation of which is directed at the perception, understanding and the conscious reflection of the essence of the objects of knowledge in one's own knowledge formed as a result of operating educational-cognitive skills (Golodiuk[19], 2013);
- key elements of the educational-cognitive activity are *the subjects* (the teacher, students), *objects* (objects of material, reified, symbolic, and mixed forms of representation of mathematical content), *cognitive activity* aimed at the cognition of an object (reproducing, creative and non-situational methods of the manifestation of cognitive activity), *means* (means, represented by the types of activity (perceptive, problem-focused, search, variable, activity-observation, heuristic, research, training research, design, graphic, practical, and

[18] Miier T. I. (2016) *Organization of educational and research activity of junior pupils. Monograph.* Kirovograd, Ukraine: FO-P Aleksandrova M. V. (in Ukr.).

[19] Golodiuk, L. S. (2013). Educational and research activity at the mathematics lessons as a form of the formation of learning and research skills of secondary school pupils. *Science and Education a New Dimension: Pedagogy and Psychology*, 6, 7–10. (In En.).

activity modeling), and *techniques* that serve as a basis for establishing the interaction between the subjects of the educational-cognitive activity (task assignment and communication); *the results of activity* (the realization by the students of the independent, conscious and effective cognition of mathematical objects through the performance of cognitive acts; the transition to the highest level of a math teacher's readiness to organization of students' educational-cognitive activity of in the 5^{th}–9^{th} grades; the transition to the highest level of students' readiness to the implementation of learning activities in class and when the classes are over);

- the purpose of students' educational-cognitive activity is recognized as self-sufficient education, the prediction of the main expected result of the implementation of this type of activity in main school is interpreted as the cognition of mathematical objects on the basis of the elaboration and implementation of a cognitive act in various types of educational-cognitive activity;
- the purpose of the organization of the students' educational-cognitive activity in main school mathematics class is defined as the guidelines setting the direction of the movement to the intended result. Regarding a teacher's activities, the objectives specify the following: at what stage of the lesson, during which activity the teacher will develop the knowledge and (or) skills; what means will contribute to children's performing educational-cognitive activities. Formulating the objectives by students serves as the conscious focus on knowledge and/or skill that is being formed or activated in a cognitive action;
- the content of the educational-cognitive activity of students in main school in learning mathematics is:

1) a pedagogically adapted system of knowledge, abilities and skills that provide the students' educational-cognitive activity;
2) a process;
3) a result of purposeful, pedagogically organized and systematic involvement of secondary school students

into the actualization of the educational-cognitive activities while learning mathematics.

Given the above, to the substantial component of the math teacher's preparedness to the organization of students' educational-cognitive activity in the 5^{th}–9^{th} grades the following types of knowledge grouped in the following units have been referred:

- *knowledge of the process, which is carried out with the aim of the implementation by students of the unassisted, conscious, and effective cognition of the mathematical objects on the basis of performing a number of cognitive acts*, includes the following knowledge:
  - knowledge about the essential features of the educational-cognitive activity;
  - knowledge about the essence of the phenomenon «organization» as a process carried out with a specific educational objective;
  - knowledge about the essence of the phenomenon «organization» as a process of management;
  - knowledge about the essence of the phenomenon «organization» as a process of control;
  - knowledge about the characteristics of students' educational-cognitive activity in their early teenage years;

- *knowledge about the subject of educational-cognitive activity, the role of which is performed by a student or students as a collective subject,* is the
  - knowledge of the psycho-physiological features of the development of junior and senior schoolchildren;
  - knowledge about the peculiarities of the organization of their activities in math class, knowledge about the mental abilities and cognitive styles of students and their influence on the process of the organization of students' educational-cognitive activity when learning mathematics in main school;
  - knowledge about mathematical giftedness;

- *knowledge about the objects that can be used to organize students' educational-cognitive activity* is the

  - knowledge about the objects of material forms of representation of mathematical information;
  - knowledge about mate-rial objects;
  - knowledge about objects, symbolic forms of the presentation of mathematical content;
  - knowledge about the features of the mixed (material, materialized, and symbolic-character) forms of the presentation of mathematical content;
  - knowledge about the characteristics of the students' perception of the objects of the material and materialized symbolic and mixed forms of the representation of mathematical content;

- *knowledge about student's cognitive activity aimed at cognizing the object* (knowledge about the specific features of actualizing the manifestation method of students' cognitive activity; knowledge about the peculiarities of the creative manner of the manifestation of students' cognitive activity; knowledge about the peculiarities of the situation-independent way of manifesting students' cognitive activity);
- *knowledge about the means that: can be used to organize students' educational-cognitive activity in math class* is the

  - knowledge about the nature and peculiarities of the organization of perceptual activities;
  - knowledge about the essence and peculiarities of the organization of such components of perceptual activities as perceptual-cognitive, perceptual-descriptive, perceptual-recogni-zing, perceptual-symbolic, perceptual-analytical, and perceptive-selective activity;
  - knowledge about the essence and peculiarities of the organization of problem-oriented activities;
  - knowledge about the essence and peculiarities of the organization of search activities;
  - knowledge about the essence and peculiarities of the organization of variable activity;

  - knowledge about the essence and peculiarities of the organization of observational activity, knowledge about the essence and peculiarities of the organization of heuristic activity;
  - knowledge about the essence and peculiarities of the organization of research activity;
  - knowledge about the essence and peculiarities of the organization of educational and research activity;
  - knowledge about the essence and peculiarities of the organization of project activities;
  - knowledge of the essence and peculiarities of the organization of graphic activity;
  - knowledge about the essence and peculiarities of the organization of practical activities;
  - knowledge about the essence and peculiarities of the organization of simulation;

- *serve as the basis for building up the interaction between the subjects of educational-cognitive activity* – knowledge about the nature and characteristics of the use of communication as a key tool and basis of the organization of educational-cognitive activity of students at the lessons of mathematics in main school; knowledge about the nature and characteristics of the use of objectives as auxiliary means and basis of the educational-cognitive activity of pupils at the lessons of mathematics in main school;
- *knowledge about communication as an interaction of the participants of educational-cognitive activity* – knowledge about communication as an interaction of the participants of educational-cognitive activity, which is aimed at harmonizing and integrating their efforts with the aim of establishing relationships and achieving the common result;
- *knowledge about the result of activity* is the knowledge of what is the result, what criteria and indicators set the level of students' preparedness for educational-cognitive activity in the 5^{th} -9^{th} grades; what criteria and indicators determine the level of students' readiness to implement learning activities in class and extracurricular time.

The above blocks of knowledge define the content of the component of math teacher's readiness to arrange students' educational-cognitive activity in the 5th–9th grades. Each block of knowledge is significant and thus is considered as a necessary part of a content component.

The procedural component of the teacher's readiness to organize students' educational-cognitive activity in the 5th–9th grades is related to skills. Focusing on these mental formations, we explain the key ideas that were highlighted in the analysis of scientific sources in the context of scientific research.

Firstly, this is an interpretation of the concept of «skill» as «the method of performing actions on the basis of acquired knowledge and skills» (Brief dictionary of the Ukrainian language[20], 2010); « method of performing actions that is internalized by the subject and is supported by a combination of acquired knowledge and skills and the ability to perform actions not only in a usual environment, but in a changed one as well» (Goncharenko[21], 1997).

Secondly, the clarification of the content of skills. T. Miier[22] (2016) notes that the *skill* in its composition contains the subject knowledge (views, concepts, facts, and opinions); knowledge about the ways of acting, knowledge about the content and sequence of actions (rules, techniques); knowledge of the standards of interpersonal interaction; knowledge about the application of equipment (instruments, tools), knowledge about the rules for handling them. It also includes skills and previously acquired experience.

The establishment of the list of skills that, taken together, constitute the procedural component of the math teacher's preparedness to organize students' educational-cognitive activity in the 5th–9th grades, in our opinion, should be built in accordance with the pre-defined list of knowledge components

[20] *Brief dictionary of the Ukrainian language.* (2010). Kiev, Ukraine: ECC Prosvita. (in Ukr.).

[21] Goncharenko, S. U. (1997). *Ukrainian pedagogical dictionary.* Kiev, Ukraine: Lybid. (in Ukr.).

[22] Miier T. I. (2016). *Organization of educational and research activity of junior pupils. Monograph.* Kirovograd, Ukraine: FO-P Aleksandrova M. V. (in Ukr.).

that made up the contents of a content component of the readiness under analysis. Based on the above-mentioned information, the following skills are referred to a procedural component of the math teacher's preparedness to organize students' educational-cognitive activity in the 5th–9th grades:

- *the ability to organize the process that is carried out to actualize the students' unassisted, conscious and effective cognition of mathematical objects through performing a number of cognitive acts* – the ability to formulate the goal and objectives of one's activity; the ability to control the interaction of subsystems that act as the subject and object of management; the ability to manage students' activities through the use of direct or indirect means, asking additional questions, which coordinate the course of the arguments, use of cards-instructions, diagrams, tables, algorithms, regulations, etc.;
- *the ability to organize educational-cognitive activity taking into account the acquisition of the status of «a subject» by one student or by a pair of students (group, class) – "a collective subject"; based on the psychophysiological features of students of younger and older adolescent age, different cognitive styles, intellectual abilities and taking into account the mathematical giftedness* – the ability to organize educational-cognitive activity unassisted, in a pair with another student, in group and class; the ability to organize educational-cognitive activity of adolescent students; the ability to organize educational-cognitive activity of teenage students; the ability to organize educational-cognitive activity of students taking into account their cognitive styles, intellectual abilities and mathematical giftedness;
- *the ability to organize students' educational-cognitive activity through the use of the objects of material, reified, symbolic, and mixed forms of the presentation of mathematical content* – the ability to organize educational-cognitive activity of students through the use of the objects of the material form of presenting the mathematical information; the ability to organize students' educational-cognitive activity through the use of the objects of the

materialized form of the presentation of mathematical information; the ability to organize educational-cognitive activity of students through the use of sign and symbolic forms of the presentation of mathematical content; the ability to organize educational-cognitive activity of students through the use of the mixed (material, materialized, symbolic) forms of the representation of mathematical content;

- *ability to organize students' varied cognitive activity during the implementation of educational-cognitive activities* – the ability to compile (to select) the tasks and (or) the assignments to actualize the reproductive method of the manifestation of cognitive activity of students; the ability to compile (select) the tasks and (or) the assignments to actualize the creative method of the manifestation of students' cognitive activity; the ability to compile (select) the tasks and (or) the assignments to actualse the situation-unrelated method of the manifestation of the students'cognitive activity;
- *ability to select and use the means affectively: to organize students' educational-cognitive activity in learning mathematics* is the ability to organize perceptive activities; ability to organize perceptive-cognitive activity, ability to organize perceptive-descriptive activities, ability to organize perceptive recognizable activities, ability to organize perceptive-symbolic activities, ability to organize perceptive-analytical activity, the ability to organize perceptive-selective operation; the ability to organize problem-oriented activities; the ability to organize search activities; the ability to organize variational activities; the ability to organize observative activities; the ability to organize heuristic activities; the ability to organize research activities; the ability to organize learning and research activities; the ability to organize project activities; the ability to organize graphic activities; the ability to organize practical activities; the ability to organize modeling activities;
- *as a basis for building up the interaction between the subjects of educational-cognitive activity* – the ability to build communication as the basis of the educational-

cognitive activity of students in a math class; the ability to organize students' educational-cognitive activity a math class using such means as tasks and assignments;
- *ability to organize interpersonal interaction* – the ability to maintain the subjectivity of students while interacting with them, the ability to use communication as a means of uniting the efforts of the participants of educational-cognitive activity with the aim of establishing relationships and achieving the common goal;
- *the ability to carry out the activities of reflection, reflection of content of educational material and reflection of emotional states, the ability to analyze performance* – the ability to perform the understanding of the practices (activities), techniques of working with educational material (reflection on the activities); the ability to clarify how students understand the course material (reflection on the content of educational material), the ability to actualize the appropriate management of an object as a management subsystem based on one's one's own emotional state and the emotional response of students to the process of the organization and actualization of educational activities; the ability to analyze intermediate and final results of one's own activities on the organization of educational-cognitive activity of students; the ability to analyze intermediate and final results of the students' implementation of this activity.

Therefore, the readiness of math teachers to organize students' educational-cognitive activity in the 5^{th}–9^{th} grades as a holistic formation includes motivation, content and procedural components, which, taken together, ensure the effective organization of students educational-cognitive activity in the 5^{th}–9^{th} grades in the learning process in a math class and extracurricular time. Components of indicated readiness are represented by a set of criterion of indicators. Let us establish the correspondence between them in the following way.

1. *The motivational component*: cognitive and professional motives.

2. *A substantial component*:

1) knowledge about the process, which is being carried out with the aim of the actualization by students of the independent, conscius and effective asquisition of the knowledge about the mathematical objects on the basis of performing a number of cognitive acts; (further knowledge about the process…);
2) knowledge about the subject of educational-cognitive activity, the role of which is performed by a student or students as a collective subject (thereafter – the knowledge about the subject…);
3) knowledge about the objects that can be used to organize educational-cognitive activity of students (thereafter – knowledge about the objects…);
4) knowledge about the tools that can be used to organize educational-cognitive activity of students in learning mathematics and is the basis for building up the interaction between the subjects of educational activity (further – the knowledge about resources…);
5) knowledge about students' cognitive activity aimed at the asquisition of the knowledge about the object (thereafter – knowledge about cognitive activity…);
6) knowledge about communication as an interaction of participants of educational-cognitive activity;
7) knowledge about the result of activities (thereafter – the knowledge about the result…).

3. *Procedural component*:
1) *the ability to organize the process* to let students understand and cognize *mathematical objects on the basis of performing a number of cognitive acts*);
2) the ability to organize educational-cognitive activity taking into account the acquisition of the status of «subject» by one student or a couple of students(group, class) – the status of a collective subject, taking into account psychophysiological characteristics of students of younger and older adolescent ages, different cognitive styles, intellectual abilities and taking into account the mathematical giftedness;

3) the ability to organize educational-cognitive activity of students through the use of objects of material, reified, symbolic, and mixed forms of presentation of mathematical content;
4) *the ability to organize varied cognitive activity of students during the actualization of the educational-cognitive activity;*
5) the ability to effectively select and use the means for the organization of the educational-cognitive activity of students in learning mathematics and the means that serve as the basis for building up the interaction between the subjects of educational-cognitive activity;
6) the ability to organize interpersonal interaction;
7) the ability to carry out the activities of reflection, reflection of content of educational material, reflection of emotional states of the subjects of the activity and the ability to analyze the results.

Conclusions

Organization of students' educational-cognitive activity is supported by teacher's research and student's behavior. In the case of age or life crises, the process under study is implemented on the basis of the domination of integration, that is – students unite in groups where the interaction would be comfortable. In the absence of age or life crises this process is based on the domineering of differentiation (separation) as a conscious choice of each student of one of the types of differentiation, in particular –

- differentiation in the manifestation of cognitive activity in a different way («I want»);
- differentiation by the desire to develop their abilities («I»);
- differentiation in the manifestation of cognitive activity in a different way and the desire to develop their abilities («I can and I want»);
- differentiation in the manifestation of cognitive activity in a different way and the desire to develop the ability of Others («I can and want to help Others»).

Organization of students' educational-cognitive activity in learning mathematics involves communication between the participants in such systems:

- «a teacher as a direct subject – students as a collectivesubject – a mathematical object of cognition»;
- «a student / student as an individual-collective subject – a mathematical object of cognition – a teacher as an indirect subject»;
- «a student as an individual subject – groups of students as collective subjects – a mathematical object of cognition – a teacher as a directly-indirect subject»;
- «a student as an individual subject – a mathematical object of cognition – a teacher as an indirect subject».

Pedagogical conditions for the organization of students' educational-cognitive activity in learning mathematics in class and extracurricular time are interpreted in the context of the phenomenon of «readiness» ("preparedness") as a holistic formation that includes motivation, content and procedural components, which together provide the effective activity of the subjects of educational-cognitive activities (of a teacher and students).

1.2. Topic-oriented Upgrade of Subject-oriented Educational System

Borislav Lazarov

Introduction

The secondary school educational system in Bulgaria is subject-oriented. That means the students study separately languages, mathematics, science, history etc. and any subject has its own specific educational goals. Such school system was rather effective in a sustainable planned economics that used to be the Bulgarian one 30 years ago. Nowadays the economic situation dramatically changed. Business dictates new rules to the Bulgarian labor market, or more precisely, no rules at all. The individual success is conditioned mainly by a wide range of skills and knowledge that are expected to be transferable and applicable in the context other than the school one, where these skills and knowledge were initially formed. In short: deep analytical knowledge provides fewer opportunities for social success than a large package of skills in various areas. Hence, the goal that stands on agenda in Bulgarian education today should be reformulated: not a bunch of key competences but a kind of synthetic competence to be formed as the output of the secondary school.

It is plausible the Berryman's view point [as quoted by (Wicklein, 1997, p. 73)[1]] that "individuals do not predictably use knowledge learned in school in everyday practice, nor do they use everyday knowledge in school settings. Perhaps most importantly, learners do not predictably transfer learning across school subjects". If it is so, then the paradigm of the compulsory education should include not only the development of students' knowledge and skills but also these knowledge and skills should be multifunctional and transferable in both directions (school) ↔ (everyday experience).

[1] Wicklein, R. (1997). Curriculum Focus for Technology Education. *Journal of Technology Education*, 8(2), Spring.

Our standing point is that incorporation of topic-oriented approaches in subject-oriented educational system could further the effectiveness of the secondary school education.

Terminology notes

We talk about topic-oriented education when a particular object or phenomenon is studied from several perspectives by applying different analytical approaches. For instance, when the Earth is examined from the viewpoints of astronomy, geology, climate, biosphere etc. in an integrated educational process. The working concepts we are going to use further in this chapter were introduced by us in several publications to describe the structural parts of some didactical models (Lazarov[2], 2010; Lazarov[3], 2013; Lazarov, & Severinova[4], 2014).

1. We consider the local behavioral environment (LBE) of a student as a complex socio-economic and cultural structure including:

- people related to the student's behavior (teachers, parents, classmates etc.);
- institutions that organize education and creative work (school, clubs etc.);
- events that provide opportunities to manifest the achievements (tournaments, conferences etc.);
- systems of values that form the cultural context of the student (motivation factors, anticipation about the future professional realization etc.).

2. The synthetic competence is a multifunctional and transferable complex of knowledge, skills, and attitude (KSA) which enables the individual to meet the challenges of

[2] Lazarov, B. (2010). Building Mathematics Competence via Multiple Choice Competitions. *Journal of the Korean Society of Mathematical Education*. Vol. 14. No. 1, 1-10.

[3] Lazarov, B. (2013). Application of some cybernetic models in building individual educational trajectory. *Information Models and Analyses*. Vol. 2, No1, 2013, 90-99.

[4] Lazarov, B., & Severinova, D. (2014) Incorporating integrated approach in secondary school. *MEST Journal*, 15 07, 2(2), 2014, 113-120, doi:10.12709/mest.02.02.02.12.

their LBE successfully. It is also age-sensitive, which means that it develops along the individual's growth.

3. By individual educational trajectory (IET) we understand the organizational frame and plan for realization of a medium term educational process that is coherent with the individual specifics of the learner and provides opportunities for the optimal development of his/her creative potential.

4. We determine the integrated approach in education as a way for mutual consideration of interdisciplinary topic via the application of the methodologies of several subjects that pursuit a multifaceted educational goal. Any particular methodology acts in its own domain and uses its own standards and evaluation. Students are urged to apply their knowledge and skills in a new context and as a result, they are expected to develop a synthetic competence.

5. The project-oriented initiative (POI) is an interdisciplinary educational activity aimed at considering a particular topic from different perspectives via an integrated approach. Our experience refers to the POIs in which mathematics component occupies the central position.

6. We have constructed an administrative and didactical tool called processing matrix (ProM) to technologize the implementation of the POIs when applying the integrated approach. It includes a comprehensive list of didactical and administrative resources and instructions that clarifies the role of all individuals involved in a particular POI as well as the interaction between them.

Preconditions and motives for topic-oriented education

There exists a solid fundament for developing students' synthetic competence in Bulgarian secondary school and this is the organization in the primary education. Integrated approach is applied in the primary school as the most natural way of introducing concepts and elaborating them in different educational contexts. The pupils have a general idea for the world and society as organic unity and the school is the place where they get systematically more details about the things and relations between them mainly from their teacher.

However, crossing the border between primary and seconary school the world's big picture disintegrated into physics, chemistry, math, history etc., and the things and phenomena are studied separately from the viewpoint of the different school subjects. The knowledge is presented analytically and the output goal of the secondary school is declared as eight key competences to be formed[5] (European Commission, 2006). In fact, the individual education stops at the analytical level according to the classical Bloom model[6] (Bloom et al., 1956).

Our standing point is that the secondary school should provide at least one step further and we claim synthesis as a desired level of output knowledge. Developing the idea of competences, we introduced the concept of synthetic competence as more coherent with the basis for personal success in the modern society. The main feature of the synthetic competence is the decontextualization[7] (Lazarov, 2014), i.e. that the package KSA, built in a clinical educational context, should be transferable in another (significantly different) context.

A brief review of pre-integrated approaches

The integrated approach is one of the possible ways to reach the goal declared above – building synthetic competence. The integrated education has its roots deep in the history but also a solid ground in modern times. For instance, Pythagoras applied the mathematical methods in studying harmony; he associated with numbers even some abstract concepts as justice[8] (Van der Waerden, 1968). The benefits of integrating

[5] European Commission (2006). *Recommendation of the European Parliament and of the Council*. Retrieved from http://formacion.educalab.es/eva2013/pluginfile.php/2331/mod_resource/content/2/Key%20competences%20for%20lifelong%20learning.%202004. (active in Sep 2017)

[6] Bloom, B., Engelhart, M., Furst, E., Hill, W., & Krathwohl, D. (1956). *Taxonomy of educational objectives: The classification of educational goals*. Handbook I: Cognitive domain. New York: McKay Company.

[7] Lazarov, B. (2014). Decontextualization. *Mathematics and Mathematics Education. Proceedings of the 43rd Conference of the Union of Bulgarian Mathematicians*. Bulgaria, Borovetz, 2-6 April 2014. 67-77. (in Bulg.)

[8] Van der Waerden, B. (1968). *Probuzhdashta se nauka*. Sofia, Bulgaria: Nauka i izkustvo. p132 (in Bulg.)

mathematics, science and technology education were recognized long ago (Adelman[9], 1989; LaPorte, & Sanders[10], 1993). These subjects were also unified in the European Commission's working group WG3 (active in 2002-2004) to elaborate a common strategy and to develop educational programs for European member states[11] (Lazarov, 2004).

The various forms of curriculum integration can be jointly considered starting with discipline-based options where the subjects act separately. The first step towards integration could be formation of parallel disciplines where the content in the particular disciplines is mutually compatible. The next steps in the evolution of this idea are multi-disciplinary thematic approaches where the various subjects contribute to a central theme and interdisciplinary concepts and topics where discipline concepts are chosen because of their direct relevance to the theme[12] (C.J.Marsh, as quoted by Venville et al. 1998). In 1990's the integrated approach was adopted as global educational strategy in three US counties: Colorado, Nebraska and Oklahoma and the effect is remarkable[13] (Wicklein, & Schell, 1995).

In the first decade of the XXI century, another wave of integrated educational practice was connected with so-called STEM that stands for science, technology, engineering and mathematics. The UK epiSTEMe project started in 2008[14] (ESRC, 2013). Recently there appear a new trend in US to extend STEM to STEAM education by adding A (rts).

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[9] Adelman, N. (1989). *The case for integrating academic and vocational education*. Washington, DC: Policy Studies Associates, Inc.

[10] LaPorte, J., & Sanders, M. (1993). *Integrating technology, science, and mathematics in the middle school*. The Technology Teacher, March, 17-21.

[11] Lazarov, B. (2004). *European program Education and Training 2010, mathematics, science and techmology*. NIO, Sofia 2004. (in Bulg.)

[12] Venville, G., Malone, J., Wallace, J., & Rennie, L. (1998). *The Mathematics, Technology and Science Interface: Implementation in the Middle School National Centre for School Science and Mathematics*. Curtin University of Technology, Perth.

[13] Wicklein, R., & Schell, J. (1995) Case studies of multidisciplinary approaches to integrating mathematics, science and technology–education. *Journal of Technology Education*, Vol. 6, No. 2.

[14] ESRC (2013). Retrieved from http://www.educ.cam.ac.uk/research/projects/episteme/ (active in Sep 2017)

The idea is to enhance students' creativity by activating the whole brain capacity[15] (White, 2015). Integrated (multi-disciplinary, phenomenon) education is also declared as national strategy in Finland[16] (Halinen, 2015), which started in 2016.

Close to the scope of the integrated education is an initiative in Russia related to so called metapredmetnost (i.e. meta-subjects), which is replica of the integrated approach[17] (Gromyko, & Polovkova, 2009). The drama education is also an integrated educational technology. It could be incorporated in the classroom teaching[18] (Kelner, 1993) or realized independently as extracurricular activity[19] (Lazarov, & Kokinova, 2014).

The Bulgarian experience

The Problem Group on Education, which was founded and headed by Blagovest Sendov, made a significant leap in the integrated education in 80's of XX century. The topics in the textbook *Ezik i matematika za Parvi Klas* (Language and Mathematics for the 1st Grade) were designed as a conglomerate of themes in mathematics and Bulgarian language with parallel columns in English and Russian[20] (Sendov&Novachkova, 1985). In this case, we can speak about an integral approach, which means a general methodic in teaching different subjects, which turns them into one new meta-subject.

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15 White, H. (2015) *Our education system is not so much "broken" – as it is totally outdated!* Retrieved from http://steam-notstem.com/articles/our-education-system-is-not-so-much-broken-as-it-is-totally-outdated / (active in Sep 2017)

16 Halinen, I. (2015). *What is going on in Finland? – Curriculum Reform 2016*. Retrieved from http://www.oph.fi/english/current_issues/101/0/what_is_going_on_in_finland_curriculum_reform_2016 (active in Sep 2017)

17 Gromyko, N., & Polovkova, V. (2009). *Metapredmetnyj podhod kak iadro rossijskogo obrazovaniia*. Retrieved from http://www.teacher-of-russia.ru (active in Sep 2017) (in Rus.)

18 Kelner, L. B. (1993) *The Creative Classroom: A Guide for Using Creative Drama in the Classroom PreK-6*. Netherland. Heinemann, Portsmouth.

19 Lazarov, B., & Kokinova, S. (2014). Developing Components of Synthetic Competence via Drama Education. *Obrazovanie & Tehnologii,* Vol. 5. (in Bulg.)

20 Sendov, Bl., & Novachkova, R. (1985). *Language and mathematics for 1st grade*. PGO, Sofia. (in Bulg.)

The integral approach has been developed independently parallel to the Problem Group on Education by a team of researchers in humanitarian subjects. Seven basic skills were highlighted as integral structures in curriculum [21] (Penkova, 1990). One can easily recognize them in modern PISA descriptions of attitude[22] (PISA, 2007).

Further development of the integral approach was planned to be done by incorporating computer corners as relatively independent units in computer programming[23] (Lazarov, 1991) but in early 90's the Problem Group on Education was terminated. A project about integrating the Divine Commedia and astronomy[24] (Tabov et al., 1999) could be considered as a reminiscent of that style, but (as far as we know) it was not operationalized into regular school practice.

Obstacles and opportunities

Integrated approach is difficult to be implemented at national level in Bulgaria. The Bulgarian legislation allows extracurricular activities to become a part of the official school strategy. However, at this level the situation leaves much to be desired. Among the main obstacles, we will point the following: lack of national standards and official methodical instructions, as well as lack of specially prepared teachers.

The Project *Uspeh* (that means Project Success) of the Bulgarian Ministry of Education and Sciences used to be a platform for some pilot didactical experiments in 2007–2015. One can read among the goals of this Project the following lines:

- students' additional knowledge, skills and competences to be developed;

[21] Penkova, R. (1990) Language subjects. In Kotova, I. (edt) *Integral structures in secondary school curriculum content*. Narodna Prosveta, Sofia, 60-61 (in Bulgarian)

[22] PISA (2007). *The definition and selection of key competencies.* https://www.oecd.org/pisa/35070367.pdf (active in Sep 2017)

[23] Lazarov, B. (1991). An Idea of Using Computer in Mathematics Education. A talk given at BISME-2, Bratislava, 23-25 Aug 1990. Abstract is published in the *Proceedings of BISME-2*.

[24] Tabov,J., Muirhead, J., & Vassileva, A. (1999). Dante and the humanities. *The Teaching of Mathematics*, Vol. II, 1

- students' free time to be fulfilled by meaningful activities taking into account their personality[25] (MON, 2007).

Support of the extracurricular activities in the frame of the Project *Uspeh* was in several directions: legitimating extracurricular activities as a part of school plan; financial support; documentation center etc. A large number of teachers developed their initiatives in the frame of the Project and some good practices took place. However, the integrated approach requires the school level of management and here was the most important problem: missing attitude of the majority of the school principals to step on this way. An explanation for this is that the tradition in education is very important and the shortage of good examples is serious restraint for the school administration to adopt any form of integrated approach.

The new Bulgarian secondary school law defines a new schools status: innovative school[26] (BG, 2016). These schools are given a large autonomy to manage the school plan, to arrange the curriculum, and to apply innovative approaches (ibid, p 14). Further, the compulsory syllabi as well as the achievement standards operate with the key competences and the implementation of the integrated approach meets officially the frame of the key competences[27] (MON, 2016). It means that the compulsory classroom education will stay more or less at the analytical level. To resolve this problem the Bulgarian legislators introduced the option: schools could organize interdisciplinary education in so-called integrative subjects[28] (MON, 2015). Thus, the level synthesis in Bulgarian educational system becomes an area of the particular school strategy.

[25] MON (2007). Bulgarian Ministry of Education and Sciences. *Project BG051PO001-4.2.05-0001*. Retrieved from http://uspeh.mon.bg

[26] BG (2015) Republic of Bulgaria. *Zakon za Preducilistnoto i Ucilistnoto Obrazovanie* (*The law about the pre-school education and the school education*), Darzhaven Vestnik, No 79, 13.10.2015, part 2, catch 38 (7). (in Bulg.)

[27] MON (2016). Bulgarian Ministry of Education and Sciences. *Naredba_11_01.09.2016_ocenjavane*. (in Bulg.)

[28] MON (2015). Bulgarian Ministry of Education and Sciences. *Naredba* 5 *30.11.2015 obshtoobr_podgotovka 1*. (in Bulg.)

Searching a compromise

An idea of the curriculum that meets educational standards and at the same time is the integrated one is given in[29] (Drake&Burns, 2004). Two types of the integrated curriculum alignment of standards are under consideration: external and internal. The first type reflects the mandatory requirements while the second ones are context oriented. Waldorf school practice gives us another example of the curriculum internal orientation and the whole school integration. Usually the main lesson ties one topic to as many disciplines as possible, i.e. an integrated approach is implemented. Since Waldorf schools are autonomous institutions, they are not required to follow a prescribed curriculum[30] (Woods, & Woods, 2006), even if they are state-funded. The situation with the Bulgarian innovative schools is similar to the both examples given above: there is a mandatory part related to the educational standards but there is also enough space for variations in classroom practice and also in extracurricular activities, which are parallel to the school program. Some of these schools started the practice of the whole school integrity. Such organization provides extra time for regular classes, so an integrated approach could be applied there in the intermediate mode of facultative classroom practice. Moreover, the relative independence of an innovative school in program design allows of some topics and activities, which are complementary to the main curriculum, being incorporated in the general school plan via the interdisciplinary project-oriented initiatives. In fact, the status of the innovative schools provides an opportunity to evade the frame of the subject-oriented educational system.

How to catch this opportunity

The project-oriented initiatives (POI) are a powerful didactical instrument for implementing integrated approach

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[29] Drake, S. & Burns, R. (2004). *Meeting standards through integrated curriculum*. Association for Supervision and Curriculum Development.

[30] Woods, Ph., & Woods, G. (2006). In Harmony with the Child: the Steiner teacher as co-leader in a pedagogical community. *FORUM*, 48(3), 317-328.

on school level[31] (Lazarov & Severinova, 2014). In this section, we are going to share our experience in upgrading the subject-oriented school practice by incorporating some topic-oriented POI applying integrated approach. The methodology we used is based on the TRIZ model proposed by Genrih Altschuler in 1964. This model is designed for management of implementing an invention – algorithm of the innovation[32] (Al'tshuller, 1999).

The first stage of the Altschuler's model: clarifying the problem

Continuing the chain of ideas in section 3, we point two important needs that should be satisfied in the beginning of the secondary school:

- to keep the big picture of the world as united as possible;
- to form equips of professionals who can carry out the interdisciplinary education.

Our viewpoint is that if these two needs are met successfully than the educational process will be continuous and the package of KSA can be turned into competences more smoothly.

The next two stages

Here we skip the details about the second stage, which refers to the specification of the required resources, but we return to this stage later.

The third stage of the Altschuler's model is Analysis. The desired output should be determined in this stage, as well as the factors, which prevent this output from being obtained in full scale. Our new paradigm locates the integrated education as an upgrade to the traditional classroom style, i.e. the project-oriented forms are auxiliary to the traditional ones. However, our expectations are bigger. We see the outcome decomposed in three directions:

[31] Lazarov,B., & Severinova, D. (2014) Incorporating integrated approach in secondary school. *MEST Journal*, 15 07, 2(2), 2014, 113-120, doi:10.12709/mest.02.02.02.12.

[32] Altshuller, G. (1999). *The Innovation Algorithm (TRIZ, systematic innovation and technical creativity)*. Worcester, Massachusetts.

- for the students: to build the basis of a synthetic competence, relevant to their age, interest and abilities. This includes providing opportunities to any student to manifest his/her best in solving complex problems individually and in teamwork.
- for the teachers: to encourage their aspiration for innovative teaching and to support their professional growth. In addition, being part of a team, the teachers are given the chance to participate in conferences; they are urged to prepare publications.
- for the school authorities: to have a closer look on the school problems and to be engaged in solving cases. This includes sharing the success but also the responsibility.

A modification of the Altshuler's Model

The eventual obstacles to obtain maximal positive effect can be diversified as well. This step in our plan refers to the fourth Altschuler's stage – operational. The original Altschuler's model includes a matrix of antinomies where the key factors are combined in pairs and the possible contradictions are described between elements of a pair (ibid.). Our experience shows that as a rule, several factors are involved in an antinomy and the solution needs simultaneous consideration of all relations between them.

For example, let us consider the interaction between the next three segments that are crucial for orchestrating integrated education:

- processing matrix design;
- planning the individual educational trajectory;
- constituting groups with positive attitude.

The first two bullets refer to the methodic and administrative area and the third one seems to be pedagogical. It is difficult to find solution to any of them separately. On the contrary, they found more than satisfactory solution when they:

- are considered simultaneously in a particular initiative, included in the general annual schedule;

- the processing matrix of the project-oriented initiative is designed keeping in mind the specifics of any individual trajectory;
- the individual educational trajectory of a particular student is planned taking into account the other members of the team and the requirements stated in the processing matrix; the groups of students with positive attitude are formed with respect to the individual educational trajectory and the type of the initiative.

The above example shows that the 2-dimensional matrix of antinomies is better to be replaced by at least 3-dimensional one. It is impossible in practice to compose such multi-dimensional matrix, so we decided to consider any antinomy by itself. The approach we adopted was to separate the eventual antinomies in two areas of responsibility: general and particular. Problems as the theoretical frame (including school strategy), the annual plan (including time schedule and human resources) are of general type and the antinomies that appear are in the scope of the school decision-makers. The problems that appear during management of a particular initiative are in the competency of the staff responsible for it: teachers, class master, technical staff etc. Some antinomies of particular type due to more general problems (like the ones in the above example) were solved during the regular meetings of the crew. Such schema does not look technological but it is flexible and it worked. It allows realizing the entire complex of recommendations in the fourth Altschuller's stage that are related to the possible changes in the school ecology, i.e. the interaction between traditional style of teaching-learning and the individuals, involved in the integrated education.

The final stage of the Altschuler's model: synthesis

It requires estimating the effect of the integrated education on all affected structures, regulations, population. On this step of the organizational projecting, we developed and applied a control system relevant to the entire teaching-learning process. The next example will give an idea of the control types we applied on the students' individual educational trajectories.

The POI Music equations took place in 2010/2011 scholastic year with students of 5th grade[33] (Severinova et al., 2012). It included topics in mathematics, music and informatics. Students were divided into the teams of three persons each. Each team was given a set of problems which includes a part of a musical score (Figure 1.2.1). The students were expected to solve the first problems individually, but the communication inside the team was allowed. A kind of mutual monitoring was made during the discussions among the team members – the first level of control. A kind of self-control was done when the students compared the 'official' solutions with their own ones.

Figure 1.2.1 A task from the POI Music equations.

The math teacher performed the next level of control when he checked the presented solutions – this was the first control on the individual educational trajectory. It included discussions and comments on the concepts acquisition, solution routine, writing style. The individual educational trajectories were corrected by giving recommendations, as well as by stating additional problems and challenges (when it was necessary).

The software MuseScore 1.2 (Figure 1.2.2) that students used for performing the musical score did an external control. The wrong mathematical solutions or incorrect input of the notes leaded to false melody. Such mistakes gave reasons for detailed discussion on the corresponding parts of the students' work.

The final control was done during the last lesson when the teams worked together. Additional information about the effect of the initiative Music equations was collected during the math tests of the general term schedule.

[33] Severinova, D., Lazarov, B., & Benevreshka, Y. (2012). ICT supported integrated approach in mathematics and music education in 5th grade. *Obrazovanie i tehnologii*, Vol. 3/2012. 244-249 (in Bulg.)

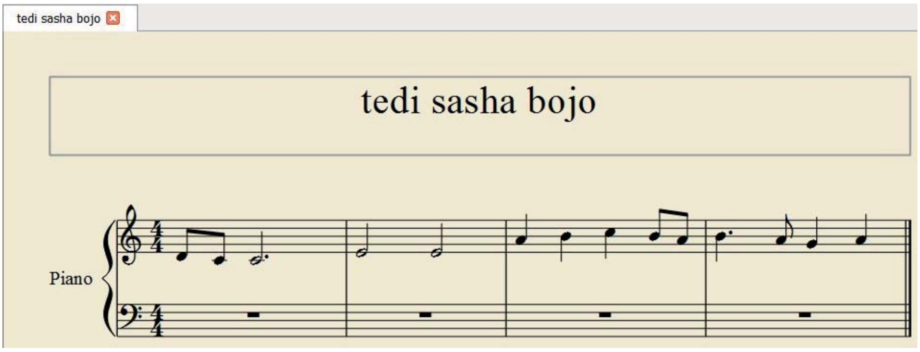

Figure 1.2.2 Solution presented as a MuseScore applet.

We cannot give an exact evaluation (in percentages) of the integrated education effectiveness but we have enough observations and qualitative assessments that form a positive big picture. The example in the next section comes to confirm such pronouncement.

A good practice

The POI Math that surrounds us took place in the second term of 2010/2011 academic year[34] (Severinova et al., 2011). It is aimed at the 5th and 6th grade students and integrated mathematics and informational technologies (which are two independent subjects in Bulgarian secondary school) to examine the KSA in a real-life context.

Students work individually or by pairs. Their task was to collect data from real-life sources (Figure 1.2.3), to form a database in electronic spreadsheets, to process the data (Figure 1.2.4) and to work out a computer presentation. Finally, the students were expected to perform their presentations at a school conference.

[34] Severinova, D., Lazarov, B., & Olejnikova, V. (2011). Integrated approach in mathematics and ICT education in 5th and 6th grade. *Obrazovanie i tehnologii*. Vol. 2/2011. (in Bulg.).

Figure 1.2.3 The data sources – the POI Math that surrounds us.

Figure 1.2.4 Processing the data from a student presentation.

This was the first time when a processing matrix was put into practice. We designed it to technologize the entire process of planning, providing and performing the initiative goals, activities and results. The processing matrix includes:

- stating objectives, indicators of progress and benchmarks;
- selection of educational content, that is to be examined;
- schedule and methodic for didactical support of the students;
- methodic for data collecting and processing the data collected;
- template for presenting information and frame for presentation design;
- assessment system;

- providing resources and organizing a school conference;
- analyzing the outcomes of the initiative and drawing conclusions.

The processing matrix allows of reconsidering the KSA that are built in the traditional curriculum topics through the perspective of their potential application in a new context. Some antinomies related to the synchronization of the time, assessment, didactic instruments etc. find solutions in the processing matrix. Its proper design allows obtaining synergetic effect from the initiative, i.e. the resulting effect of the integrated education is bigger than the effect of the expected one of any subject if it acts separately.

One can see a good practice about early business education incorporated in math curriculum in[35] (Lazarov, & Severinova, 2016). Another good practice describing an extension of the LBE in short-term and mid-term activities is given in[36] (Lazarov et al. 2015).

Concluding remarks

Since "(The) basic mathematical literacy ("numeracy") is a foundation skill for all subsequent learning in other domains of key competences"[37] (EC, 2004), the choice of mathematics as central subject in POI comes naturally. "Mathematical behavior" is about describing reality through constructs and processes which have universal application proofs (ibid.). The universality of mathematics easily mediates various subjects. The interaction of subjects in a topic-oriented project could be illustrated with a topological graph (conceptual map) in which the vertices (representing concepts) are either connected directly, or have indirect contact through

[35] Lazarov, B., & Severinova, D. (2016). Introducing business education in early middle school – math perspective. (Z. Cekerevac, Ed.) *MEST Journal*, 4(1), 65-74. ISSN 2334-7058, DOI 10.12709/mest.04.04.01.07

[36] Lazarov, B., Petrova, T., & Ivanova-Nedelcheva, A. (2015). Extension of the students' local behavioral environment via Chernorizets Hrabar educational and research program. *Obrazovanie & Tehnologii*, Vol. 5. 229-237 (in Bulg.)

[37] European Commission, Working Group B. (2004). *Implementation of "Education and Training 2010" work programme*. Key competences for lifelong learning. A European reference framework, November 2004.

mathematics. For this purpose, it is very important to break free the mathematical behavior from restriction to the quantitative methods only. Mathematics education is also charged for several basic attitudes including:

- willingness to look for reasons to support one's assertions;
- willingness to accept or reject the opinions of others based on valid (or invalid) reasons or proofs (ibid.).

All listed above reasons nominates mathematics as a natural center of the integrated approach for many POIs. Consequently, the math teachers bear responsibility in incorporating the integrated approach on school level, at least until graduate teachers appeared to carry out topic-oriented education.

The planning and execution of POI requires co-ordination of efforts of the teachers and administration. Here the school principal who is expected to have both experience and theoretical background performs a leading role. Teacher's enthusiasm is necessary but not sufficient condition for organizing POIs, as well as administrative orders alone does not help very much in realization of effective integrated education. When any of these conditions are not ensured, the implementation of the integrated approach is rather limited. However, when the efforts of a team of motivated experienced teachers are supported by the school administration and a proper theoretical model is applied, then an effective process of integrated education could happen. In this scenario, the topic-oriented activities upgrade a subject-oriented teaching to a complex educational system, which is coherent with the modern social needs.

CHAPTER 2. LEARNING MATHEMATICS AT SCHOOL

2.1. Characteristics of Concepts as Objects of Assimilation in Teaching Mathematics in Schools: Theoretical aspects

Nina Tarasenkova

Introduction

To ensure quality school math education, it is of importance for the scientists in the field of didactics of mathematics and practicing teachers to understand the peculiarities of adapting rigorous mathematical theories to the abilities of students of a certain age. In this context, the relevant reorganization of not only the mathematical content, but also of those shells into which this content is wrapped up, is important. In other words, the unity of the two components of the educational process – content and semiotic – should be the subject of didactic mathematics (Tarasenkova, 2003[1], 2013b[2]).

The first of them has been under close scrutiny of numerous studies of the researches in the field of math education (Bevz[3], 1989; Cherkasov, & Stolyar[4], 1985; Dörfler[5], 2002; Lyashchenko[6], 1988; Oganesyan, Kolyagin, Lukankin, &

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[1] Tarasenkova, N. (2003). *The theoretic-methodical principles of using of the sign and symbolic means in teaching mathematics of the basic school students* (Unpublished Doctoral dissertation). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).

[2] Tarasenkova, N. (2013, b). The quality of mathematical education in the context of Semiotics. *American Journal of Educational Research, Special issue «Ensuring the quality of higher education», 1*(11), 464-471. doi: 10.12691/education-1-11-2.

[3] Bevz, G. P. (1989). *Methods of Teaching of Mathematics*. Kyiv, Ukraine : Vyshcha shkola. (In Ukr.).

[4] Cherkasov, R., & Stolyar, A. (1985). *Methodology of teaching mathematics. General methodology*. Moscow, Russia : Prosveshcheniye. (In Rus.).

[5] Dörfler, W. (2002). Formation of Mathematical Objects as Decision Making, *Mathematical Thinking and Learning, 4*:(4), 337-350. DOI: 10.1207/S15327833MTL0404_03

[6] Lyashchenko, E. I. (Ed.). (1988). *Laboratory and practical classes on the methodology of teaching mathematics*. Moscow, Russia : Prosveshcheniye. (In Rus.).

Sanninsky[7], 1980; Roganovskiy[8], 1990; Rott & Leuders[9], 2016; Slepkan[10], 2000; Usova[11], 1986; and other). The second, semiotic component is also in the focus of research (Presmeg[12], 2016; Sáenz-Ludlow & Kadunz, 2008[13], 2011[14], 2016[15]; Salmina[16], 1988; Set'kov[17], 1997; and other). However, the essence of the unity of these components, its specific characteristics and influence on the organization of teaching mathematics, in particular in the basic school, need further research.

According to the logical and philosophical understanding of the essence, process, and the results of thinking[18] (Kondakov, 1971), there exist three forms of thinking – concept,

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[7] Oganesyan, V., Kolyagin, Yu., Lukankin, G., & Sanninsky, V. (1980). *Methods of teaching mathematics in secondary school: General methodology. Textbook. Manual for physics and math students of pedagogical institutes. 2nd ed., rev. and enl.* – Moscow, Russia : Prosveshcheniye. (In Rus.).

[8] Roganovskiy, N. M. (1990). *Methodology of teaching mathematics in secondary school*. Minsk, Belarus : Vysheysha Shkola. (In Rus.).

[9] Rott, B., & Leuders, T. (2016). Inductive and Deductive Justification of Knowledge: Flexible Judgments Underneath Stable Beliefs in Teacher Education, *Mathematical Thinking and Learning, 18*(4), 271-286. DOI: 10.1080/10986065.2016.1219933

[10] Slepkan, Z. I. (2000). *Methodology of teaching mathematics*. Kiev, Ukraine : Zodiac-Eco. (In Ukr.).

[11] Usova, A. (1986). *Formation of Scientific Concepts in Schoolchildren in the Process of Education*. Moscow, Russia : Pedagogika. (In Rus.).

[12] Presmeg, N. (2016). Semiotics as a Tool for Learning Mathematics: How to Describe the Construction, Visualisation, and Communication of Mathematical Concepts by Adalira Sбenz-Ludlow & Gert Kadunz (Eds.). *Mathematical Thinking and Learning, 18*(3), 233-238. DOI: 10.1080/10986065.2016.1184953

[13] Sáenz-Ludlow, A., & Kadunz, G. (Eds.). (2008). *Semiotics in Mathematics Education.* Rotterdam / Taipei : Sense Publishers.

[14] Sáenz-Ludlow, A., & Kadunz, G. (Eds.). (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam / Taipei : Sense Publishers.

[15] Sáenz-Ludlow, A., & Kadunz, G. (Eds.). (2016). *Semiotics as a Tool for Learning Mathematics*. Rotterdam / Taipei : Sense Publishers.

[16] Salmina, N. C. (1988). *The Sign and Symbol in Teaching.* Moscow, Russia : Moscow State University Press. (In Rus.).

[17] Set'kov, V. F. (1997). *Visual aids as the basis for understanding scientific knowledge: Ontognoselogical aspect: Author's abstract*. Yekaterinburg, Russia : The Ural A. M. Gorky State University. (In Rus.).

[18] Kondakov, N. I. (1971). *Dictionary of Logic*. – Moscow, Russia : Nauka. (In Rus.).

judgment (in its resultative sense) and reasoning (in its procedural sense). Consequently, we can differentiate three types of the objects of assimilation in school mathematics: concepts and their definitions; mathematical facts (axioms, theorems, formulas, correlations etc.); modes of operation (algorithms, heuristic schemes, rules, methods and ways of solving problems and proving mathematical statements).

In the manuscript we will dwell on the characteristics of concepts in detail.

The content component of concept formation

What is concept?

A concept is viewed[19] (Kondakov, 1971) as the form of thinking about the totality of essential and nonessential properties of a particular object (subject). Consequently, the concept is a collection of judgments, the core of which is judgments that reflect the distinctive properties of the object and of the phenomenon.

The process of identifying the essential properties of an object (subject) and separating them from the inessential properties is called the definition of a concept. The goal of the concept is to answer the question, what exactly the given object (subject) that is displayed in the concept is. To do this, it is necessary to bring its essential properties in the affirmative form. The phrase "definition of a concept" must be understood as activities concerning the definition of a concept. The results of such activities, as a rule, are expressed by a certain verbal construction, which is called the definition of the concept, or *definition*.

What is an example, a counterexample and a free object to the concept?

Defining the concept, we actually separate the set of objects, which is the volume of the generic concept, into two subsets. One of them forms the volume of the concept that is defined, and the second subset includes all other objects from the scope of the generic concept that are not related to the given concept. From the point of view of logic, the second subset is the complement of the scope of this concept in the universal

[19] Kondakov, N. I. (1971). *Dictionary of Logic*. – Moscow, Russia : Nauka. (In Rus.).

class, which is represented by the volume of the generic concept. Such an addition forms the scope of the concept, which is controversial regarding this concept.

In school mathematics, the concepts that contradict certain concepts are in some cases the main objects of assimilation (for example, irregular fractions are studied specifically at the level of definition), in other cases they carry an accompanying didactic load (for example, a nonrectangular triangle as a separate concept is not specifically studied, but determining a rectangular triangle, as a rule, it is emphasized that all other triangles are not rectangular), in other cases (and they are in the majority) the contradicting ideas are not even mentioned.

However, any knowledge can become effective only if it is formed as a positive-negative construct. Therefore, we believe that every notion of school mathematics should be revealed together with the examples from the scope of the concept that contradicts this one.

In this regard, we attach special importance to such concepts as "example", "counterexample" and "free object"[20] (Tarasenkova, 2003). An example of a concept is any object from the scope of a given concept. Giving examples (illustration of the concept) is a powerful didactic means of forming a correct mathematical concept in students.

By counterexample to the concept we mean a material or materialized object, illustrating such a formulation of the definition of this concept, in which there are some significant signs of this concept. For example, in the school definition of the circle[21] (Burda, & Tarasenkova, 2007, 2011, 2015), two specific properties can be distinguished:

a) the figure on the plane is formed by all points with a certain property;

b) points of the circle are equidistant from a certain point of this plane. In the erroneous definition of this concept, there may be no first, or second, or both species properties. So, we can formulate at least three kinds of false definitions

[20] Tarasenkova, N. (2003). *The theoretic-methodical principles of using of the sign and symbolic means in teaching mathematics of the basic school students* (Unpublished Doctoral dissertation). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).

[21] Burda, M. I., & Tarasenkova, N. A. (2007, 2011, 2015). *Geometry. Textbook for the 7th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).

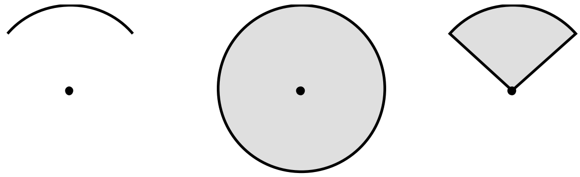

Figure 2.1.1 Counterexamples to a circle.

of the circle, and thus create at least three counterexamples. In the first case, a counterexample to the concept of a circle will be an arc of a circle, in the second – a circle, and in the third – a sector of a circle (Figure 2.1.1). Thus, counterexamples are objects from the volume of the concept controversial to the given one.

Free objects in relation to this concept are objects that go beyond the scope of the concept that is generic for it. For example, if we omit the word “figure” in the formulation of the definition of the circle, which is the name of the generic concept, we also get a false definition, which is illustrated by a free object with respect to the concept of a circle – a tree behind a window, a chair, chalk, etc. The relationship among examples, counterexamples and free objects are schematically shown in Figure 2.1.2.

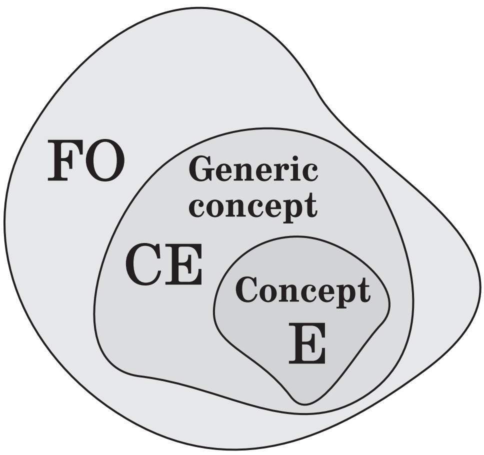

Figure 2.1.2 Relations among examples, counterexamples and free objects:

E – examples of concept,
CE – counterexamples to concept,
FO – free objects.

What are the functions of counterexamples and free objects in teaching?

In teaching mathematics counterexamples and free objects perform at least three functions. The first of these is associated with the creation of the context, against which the content features of the basic objects of assimilation are revealed more fully, become more comprehensible for the students, and thus provide a more thorough understanding of them by the students. The second function of counterexamples is the formation of constructs "positive – negative", the presence of which in the personal experience of students is the key to the effectiveness of their knowledge[22] (Tarasenkova, 2003). In their third function, the counterexamples to the concepts, facts and methods of activity are the means of adjusting the knowledge, skills and abilities of the students[23] (Tarasenkova, 2013a).

The semiotic component of concept formation

What is an object text?

To objectificate the contents of school mathematics, various semiotic and symbolic means (SSM) are used[24] (Tarasenkova, 2014). They allow of fixing – in the expanded or minimized form – the gist of separate objects of assimilation – certain concepts, mathematical facts (axioms, theorems, formulas, etc.), modes of activity (rules, algorithms, method of solving problems and proving mathematical statements) as well as their integrity – a fragment of a systematic scientific theory.

Since the concept of "text" covers two different aspects of the language fixation of mathematical content – at the level of individual object assimilation and on the level of system of knowledge, one should distinguish between these cases by

[22] Tarasenkova, N. (2003). *The theoretic-methodical principles of using of the sign and symbolic means in teaching mathematics of the basic school students* (Unpublished Doctoral dissertation). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).

[23] Tarasenkova, N. (2013a). Specifications of coding geometric concepts. *Science and Education a New Dimension: Pedagogy and Psychology*, *5*, 7-11. (In Rus.).

[24] Tarasenkova, N. (2014). Peculiar Features of Verbal Formulations in School Mathematics. *Global Journal of Human-Social science : G : Linguistics, & Education*, *14*(3), 61-67. Retrieved from http://globaljournals.org/journals/human-social-science/g-linguistics-education

entering into circulation two different terms. In the first case, when the text reflects the essence of a particular object of assimilation, it is appropriate to use the term *object-based text*. In the second case, when the text is a discrete part of a scientific theory, namely – a theme from the school course of mathematics, it is appropriate to use the term *instructional text*. It is clear that an instructional text can contain one or more object-based texts as its components.

The essence of the concept may be fixed in the object texts of two types – in the formulation of the definition of the concept or in the description of the concept, which verbally fixes the results or the progress and the results of the application of the definite methods of the disclosure of the essence of the concept.

What are the peculiar features of defining the concept as an object text?

We consider it right to differentiate strict and lax formulations. A strict formulation is a logically structured and stylistically perfect text that is constructed according to certain rules of logics and natural language. A strict formulation is devoid of redundancies and expressive content, it is concise in meaningful, it's every word being an important text component. Stylistic modifications are allowed, but limited in number. A lax formulation (a correct wording made by the pupils and students in their own words) can be not logically structured and may be stylistically imperfect. In the instructional process such kind of formulation has certain didactic functions, both – at the stage of the assimilation object introduction and at the other stages of its mastery by the students. In our opinion, didactic balanced use of lax wording along with the strict wording should be institutionalized in school practice.

General logical structure of a strict formulation of the concept definition reflects the signified concept, a generic term, species differences and the relationship between them. It can be represented by the following scheme:

the signified concept $\rightarrow$ *generic term* $\rightarrow$ *specific differences*.

However, the text which is the wording of the definition in its certain stylistic modification can be built in different ways, by inductive as well as and deductive principle. In the first case, the text reflects the verbal passage from the particular to the general, and in the second case – vice versa. For example, in textbooks, manuals on the methods of teaching mathematics, reference books the following definitions of the concept "Square" can be found:

- "A Square is a rectangle in which all sides are equal";
- "A Square is a rectangle with four equal sides";
- "A Rectangle, in which all sides are equal, is called a square";
- "If all sides of a Rectangle are equal, a rectangle is called a square".

The first two formulations of the definition of a Square directly reflect the logical structure of a general definition of the concept, thus reflecting inductive reasoning – from the signified concept to the generic concept. Therefore, they should be called *inductive formulations of concepts definitions*.

The third and the fourth formulations of the Square definition also meet the general logical structure of the concept definition, but reflect a different mental progress, which is fundamentally different from the previous one. Here deductive reasoning follows the principle – from the generic concept, the content of which is separated by some specific features of the signified concept, to the signified concept as a subspecies of the generic concept. This suggests that general logical structure is indirectly reproduced in such definition and relevant texts may be called *deductive formulations of concepts definitions*.

In the definition of a square the contents of the texts match as the square is defined through the same generic shape – a rectangle and they employ the same specific properties – the equality of all sides of a rectangle. However, semantic-symbolic components of these object-oriented texts are different – the wordings of the above definitions do not match the form of text construction. We can say that in this case the inductive and deductive formulations of the concept definition are identical in their content but semiotically different. Hence, they are to be considered to be various semantic-symbolic means of the objectification of the essence of the concept.

It is not only semiotically different but content-different definitions of the same concept which are used in school mathematics. If, for example, the concept of square is determined through a generic term "rhombus" and the related specific properties, then we will have a semantically different definition. It is clear that semiotic-symbolic components of content-different definitions can differ, although the relevant object-oriented text can be built the same way – by inductive or deductive principle.

What are the characteristics of terms used for fixing the concept?

In the school course of mathematics, two types of terms can be used – nominative and auxiliary terms. In turn, each typological group of terms can be divided into certain classes.

So, in the nominative terminology of the school course of mathematics, two classes can be distinguished. The first of these is appropriate to be called the class of general nominative terms. To the second class of nominative terminology of the school course of mathematics we refer individual nominative terms. They visualize the names of specific objects and are formed by specifying the corresponding common terms.

In the school course of mathematics, there are also terms, the sign and symbolic shell of which has the form of a nominative term with a specification, although in content they do not fix the connections between generic and species concepts. Here belong such terms as "decimal fraction", "truncated pyramid", etc. We completely share the opinion of G. Bevz[25] (2002) that such terms should be considered undivided terms.

The need to consider a group of auxiliary terms is dictated by the specifics of the organization of students' learning of the nominative terminology of school mathematics and its proper use. Auxiliary terms can serve as an additional explanation, clarification, comparison, a specific content or a mnemonic reference, directions and the like.

[25] Bevz, G. P. (2002). "Of numbers". *Mathematics in School, № 1*, 6-9; *№ 2*, 23. (In Ukr.).

What is the role of substitution activity in the process of learning concepts?

Substitutions (in the narrow sense) are activities with sign and symbolic means that represent the functional use of these means instead of the reality they designate.

The development of knowledge, skills and abilities presupposes the independent fulfillment by students of the activity of substitution. But, if the teacher did not demonstrate the basic standards of such activities, did not show examples of the improper use of substitutes, and did not disclose the consequences that such errors can lead to, then the students' correct choice of the right substitutes and their conscious use are under the certain threat. Consequently, the substitution activity that the teacher and pupils will perform during the acquaintance with the new material and at the first stages of knowledge and skills training should be based on the procedures for comparing and contrasting the "right" substitutes (examples of the objects of assimilation) and "improper" substitutes (counterexamples to the objects of assimilation). The paired use of such substitutes, in which the attention of students is fixed precisely on the opposites of their content and differences in their form, serves to prevent further mistakes.

Terminological dependence of perceptions and concepts of the students often prevents proper assimilation of nominative terminology and free command of it. Such dependence, as a rule, arises at generalization of non-essential properties of concepts. It is often provoked and reinforced by the natural "scheme of the world" of man, as well as a number of everyday terminological associations.

For example, a notorious terminological dependence of schoolchildren's ideas of an isosceles triangle – if the lateral side of such a triangle is placed horizontally, it is recognized by the students with great difficulty. A "horizontally located" isosceles triangle does not possess such visual properties, so it is not recognized. If, along with the nominative term "isosceles triangle" to use the phrase "horizontally located" or "such that it lies on the side", the recognition will be easier. Thus, the above-mentioned word combination plays the role of the auxiliary clarification terms.

It should be noted that in the example under consideration, the auxiliary terms are dependent, since, acting as separate phrases, they will not carry the completed content. They are filled with such content only in conjunction with the nominative term. In this case, we can rather talk about using a special compound term, which is appropriate to be called binary.

What is the role of coding (decoding) in the process of learning concepts?

Psychology defines coding as the activity with sign and symbolic means which lies in translating the reality (or the text that describes this reality) into the language of signs and symbols and in further decoding of the contents[26] (Salmina, 1988). Through introducing coding and decoding into school academics, it is possible to make a transition to different types of sign and symbolic expressing of the training content. Such transitions are a necessary component of theoretical thinking. They help to separate the form from the meaning and thus ensure full assimilation of the knowledge.

What are the features of coding in the process of learning concepts?

Unlike substitution, coding can be carried out not only situationally. Its main function is seen in the creation of the code structures of knowledge and operating them. We can say that encoding in this case has a specific purpose.

Therefore, the formation of code structures of knowledge and training schoolchildren in coding in its intended use should be considered as one of the main tasks of school mathematical preparation of students.

In school mathematics, each concept that is the object of assimilation for students is assigned the term – its verbal designation. In the processes of thinking, terms express the essence of those objects that they designate. They serve as the “representatives” of the corresponding meaning in consciousness[27] (Vygotsky, 1956). Therefore, the term should be considered a *verbal* or *terminological code* of the concept.

The logical-mathematical symbol is also a certain code of mathematical content. It is advisable to call it the logical-mathematical (or symbolic) code of the concept. For example,

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[26] Salmina, N. C. (1988). *The Sign and Symbol in Teaching*. Moscow, Russia : Moscow State University Press. (In Rus.).

[27] Vygotsky, L. S. (1956). *Selected psychological studies*. Moscow, Russia : Uchpedgiz. (In Rus.).

the notion of “parallel lines *a* and *b*” (terminology code) is conditionally denoted “ $a \parallel b$ “ (logical-mathematical code), the concept of “set of natural numbers” (terminology code) is denoted by the symbol “*N*” (logical-mathematical code), and the mathematical fact “point “A” has a coordinate *x* on the coordinate line” (terminology code) is denoted by the symbol “A (x)” (logical-mathematical code).

A number of concepts and facts of the school course of mathematics have a distinctive feature: together with the verbal code, they are often denoted graphically as well – with the help of geometric drawing, graphics, schemes, etc. In this case, the graphic interpretation together with the term forms a new, verbally-graphic code of the concept. Such code generally is more capacious in content and more plastic in comparison with the terminology code.

However, not every graphic interpretation of the concept can fully perform the functions of the code. Indeed, the “bare” drawing does not always contain enough information to uniquely identify, for example, the geometric concept. Metric concepts are special in this sense, for example, such concept as an isosceles triangle. Without the metric data labeled on the drawing with the help of special marks (dashes, brackets, etc.), it is not that easy to detect and it is impossible to say that this triangle is isosceles, especially when it is placed “out of the box”.

The structure of the coded (minimized) structure of the concepts (fact) includes the following components:

- *the shell of the structure* – a term for the concept (the name of the fact), together with its coded content-graphical or content-analytical interpretation, which is configured in a certain way;
- *the core of the structure* – the meaning of the graphics or mathematical sentences in which two such moments are concentrated. The essence of the first of them is that in the core structure the essential properties of the concept that are included in its definition or description are shown. The second point is related to the fact that the code-shell displays only one of the representatives of many examples of this concept;
- *the operator of the structure* – the bijection of the graphics or record and their contents.

The structure mentioned above, can serve as a single semantic unit forming the content-graphical or content-analytical code of the concept only if each of its components is formed in students, and a match between the shell and its contents is perfected to automatism. Thus, there is a need for a special technique aimed at achieving these goals.

As mentioned, coding is performed in order to further decode the content that is to correctly recognize what has been encoded by semantic-symbolic means. Decoding can be also called reading data.

Unlike verbal forms of fixing the content, non-linguistic semantic-symbolic means create a fundamentally different situation in decoding the content submitted by them. It is known that picture enters the human brain by visual analyzer channels. It is perceived simultaneously and holistically and avoids verbalization processes. The use of non-verbal semantic-symbolic means in their accompanying function of the way to illustrate the learning process is based on this.

However, in school mathematics non-verbal semantic-symbolic means can be used in the function of the independent carriers of the meaning (Tarasenkova, 2015). In this case, the verbal support of the decoding often serves for the explication of all the characteristics of the content wrapped in a non-verbal semantic-symbolic shell. In decoding non verbal data the significant role is played by visual thinking. The influence of visual thinking is particularly important in the study of geometry.

The decoding of the content wrapped in a particular semantic-symbolic shell may be otherwise called scooping the content. Here it is not always enough to have accurate and profound knowledge. One should have some experience of such activities. If this experience is formed spontaneously, it does not ensure the formation of the sufficient ground for fully-fledged independent activities of students, and thus the possibility of filling the learning process with personalized meaning and significance is limited. So, the purposeful teaching of the skills of decoding activity requires special attention in the process of mathematical training of students, which means that there is a need to create an appropriate methodology.

The stages of the concept formation

The initial stage in the process of the formation of the concept is the introduction of the concept. To introduce a concept means to reveal its essential properties, thereby separating objects belonging to the volume of the given concept, from those not belonging to it. In other words, to introduce a concept is to create a construct "examples of the concept – counterexamples to the concept" and to disclose its content.

What are the methodological schemes for introducing the concept?

In teaching mathematics they use two concept introduction schemes that reflect the nature and structure of the abstract-deductive and inductive-empirical methods. Their implementation in the educational process allows of building the procedure of the introduction of the concept through theoretical or empirical generalization respectively. The structure of each scheme traditionally identifies a number of components – individual stages of introducing the concept that differ in their objectives, the nature of mental activity of students and the manner of its organization.

Full application in math class of the concepts introduction schemes corresponding to the specific-inductive or abstract-deductive method, even with their specified schemes is a necessary but insufficient step towards creating adequate initial conditions for the formation of concepts. Equally important is the proper organization of students' activities with semantic-symbolic means (SSMA), within which the semantic-symbolic means of the reification of concepts are not any longer the objects of activity but the tools for it.

Four types of SSMA, namely — replacement, coding (decoding), schematization and simulation — being the components of learning activities of students perform a definite role in the formation of concepts. However, during the introduction of the concept depending on the chosen scheme (specific-inductive or abstract-deductive) the content of the SSMA is somewhat different.

What are the peculiarities of the initial stages of introducing the concept under a specific-inductive scheme?

The first step in the specific-inductive scheme of the introduction of the concept is connected with the students' performing of the analysis of the sensory material — the examples of the concept, its counterexamples and free objects — with the purpose of identifying the essential and nonessential properties of the concept that is introduced. So, the results of the teacher's substitution activity (performed in the course of the preparatory work) are given to the students as the ready-made substitutes. Students' activities, both — in content and form, are modeling.

The construction (choice) of an adequate sign and symbolic substitute for the concept that is introduced is in fact a construction of the concept model by ascending from its concrete example (mostly non-verbal) to the abstractly expressed essence of the concept in the primary and then in the precise definition of the concept. The construction, made by students, by the primary definition formulated in "their own words" is a new stage of modeling — constructing a verbal form of the concept model, albeit imperfect. The transition from it to an exact definition is based on analysis, comparison and contrast of content, which is represented through tangible examples in a folded, veiled form, and in the primary definition — in an expanded, verbally fixed form. As a result of the refinement of the primary definition, a perfect verbal model of the concept being formed is made.

At this stage the activity of modeling is intertwined with the activity of coding (decoding), since it is during this period that the term for the concept is introduced together with its graphic or analytical interpretation. In other words, at the stage of analyzing the sensory material and formulating the definition of the concept, the terminological and verbal-graphical (verbal-analytic) concept codes begin to form, which gradually turns into the construction of a meaningful (conceptually analytic) concept code and further into the formation of new knowledge as in corresponding construct. Here students are trained for the first time in the analysis of the text of the definition as the linguistic shell of new knowledge with the aim of translating it into another verbal (at least pictographic) or non-verbal form.

What are the peculiarities of the initial stages of introducing the concept by the abstract-deductive scheme?

During introducing the concept by an abstract-deductive scheme, meta-modeling is performed. The essence of the concept appears before the students in the form of a ready-developed verbal model of reality — the text of the definition. Reading (listening) as a decoding activity is the first stage of meta-modeling. Simultaneously the activity of conversion from the expanded verbal form to the folded form and vice versa is performed, since the definition mostly contains the term of this and the generic concept and often includes symbolic notations or graphic substitutes for the concept.

A meaningful analysis of the text of the definition is the next stage of the meta-modeling — the reorganization of the model. In order for this analysis to take place not formally, it is important to build together with it a system of examples, counterexamples and free objects in relation to this concept. In other words, for a meaningful analysis of the definition of a concept, it is advisable to combine two stages of meta-modeling — the study of the model and the interpretation of the results of the meta-modeling.

What do the two concept introduction schemes have in common?

The stage of concept illustration is common with the two concepts introduction schemes. At this stage, performing the mental activity of bringing the phenomenon under the concept, the students act according to certain rules-guidelines, and hence their activity with the SSM is a schematization. Guiding rules can be known to students and applied step by step with the reflection of each action, accompanied by their verbalization — pronouncing. In this case, it is advisable to talk about the conscious use of schematization. However, giving the examples of the concept and its counterexamples can occur at the level of the unconscious (the preconscious – the intuitive, the superconscious — the minimized, automated) application of the rules of reference.

In the first case, when the schematization occurs at a preconscious level, as a rule, students are not able to verbally substantiate their actions, because they do not know the guiding rules and act intuitively. Intuitive actions can lead both to the correct result, and to the erroneous result. With improperly intuitive implementation of the action of conceptualizing the students are often mistaken in the choice of substitutes for the concept (examples), and often refer free objects to counterexamples. Such results of the students often testify to the need for the immediate corrective interference of the teacher.

If the students do not make mistakes when citing examples and counterexamples of the concept, this may indicate both — the correct intuitive performance by the students of the action of conceptualizing (the manifestation of the preconscious) and the reduction (curtailment) of the corresponding knowledge about the scheme of activity. In the latter case, it is advisable to talk about the fact of the superconscious (minimized, automated) application of the guiding rules of the action of conceptualizing. In teaching mathematics, it is inappropriate to stop the process of mastering the concept by the students at the preconscious level.

During the demonstration of the standards for the performance of the deducing of the consequences, a new round of activity with sign and symbolic means begins, which at this stage of introduction of the concept for students is a simulation. And it's exactly how modeling needs to be organized at the lesson. In the future, in the context of conscious performance by the students of the action of deducing the consequences, this component of the application of the concept will be carried out according to the laws of the activity of schematization.

However, the introduction of the concept would be incomplete if the students were not given the opportunity to see how to change the sign-symbolic shell of the concept without changing the content of the definition or changing it. In other words, the necessary stages of introducing the concept are the stage of reformulating the definition and the stage of formulating a meaningfully different definition of the concept (if possible). At the first stage, students' activity is recoding, and at the second stage it is modeling again.

Conclusions

Taking into account the semiotic aspect and the above considerations, it is advisable to present a specific-inductive and abstract-deductive scheme for the introduction of concepts in the following form.

A specific-inductive scheme for introducing the concept

(1) On a set of special substitutes (examples of concepts, counterexamples and free objects in relation to it), they distinguish essentially general and distinctive features of these objects; separate the essential and non-essential properties of the concept; introduce the term concept.
(2) The primary (non-strict — correct, but stylistically imperfect) definition of the concept is formulated; with the help of examples and counterexamples the content of the definition as an object text is clarified.
(3) A strict definition of the concept is formulated; a meaningful analytical interpretation of the concept is constructed — its logical-mathematical symbolic designation (if any) is introduced.
(4) An illustration of the concept is performed, that is — a demonstration of the action of conceptualizing. For this, examples, counterexamples and free objects are also used, but the substitutes different from the ones used at the first stage are chosen. At the mathematics lesson, this stage often begins with the tasks like: "Give examples ..."; "Can we consider ... an example ...?"; "Why ... is not an example ...?" etc.
(5) The consequences from the fact that a particular object belongs to the given concept are inferred — they demonstrate the standard of this action and conduct the first training for the students. Often at the lesson, the following may be heard: "Here we have an example of a concept .... What follows from this? Name all the properties possessed by ...". Particular attention is paid to the fact that from the fact that a substitute belongs to counterexamples or to free objects relative to a given concept, one cannot derive the same consequences as from the example of this concept.
(6) The reformulating of the definition takes place — the replacement of one sign and symbolic shell with the

other without any changes to the content component of this object text.

(7) Meaningfully different definitions of the concept (if possible) are formulated; individual cases (if they exist) are considered.

Abstract-deductive scheme of introducing the concept

(1) A strict definition of the concept is formulated; its term and the logical-mathematical symbol are introduced.
(2) The analysis of the text of the definition is carried out — the generic concept and species properties of the given concept are identified, the nature of the links (conjunctive, disjunctive) between the properties is clarified; a concretization of the concept is made; a system of examples, counterexamples and free objects with respect to the given concept is built.
(3) The action of conceptualizing — an illustration of the concept — is demonstrated. Here, as in the scheme corresponding to the specific-inductive method, from the demonstration of the standard of performance of action, they pass to the first training attempts made by the students.
(4) The conclusions from the fact that a particular object belongs to the given concept are drawn. The specificity of drawing conclusions in the situation of choosing a counterexample or a free object with respect to a given concept, instead of its example, is analyzed.
(5) A reformulation of the definition — the replacement of one sign-symbolic shell with the other without any changes to the content component of this object text is made.
(6) Meaningfully different definitions of the concept (if possible) are formulated; Individual cases (if any) are considered.

As it has been already noted, the process of concept formation does not end at the stage of introducing the concept. When studying mathematical facts, the content of the concept is expanding being replenished with new pro→—perties. During the assimilation of the methods of activity the peculiarities of the application of concepts are revealed to the students, which becomes the prerequisite for the full assimilation of this or that concept.

From the standpoint of the approach developed by us, the following skills should be considered the criteria for the maturity of the concept: the skills of correct correlating the concept and its term — terminological encoding (decoding) the concept; correct performing the content-graphical or content-analytical interpretation of the concept, as well as recognizing the concept by its content-graphical or content-analytical codes in different situations — visual encoding (decoding) the concept; recoding the terminological code of the concept into its content-graphical or substantive-analytic code and vice versa; deploying any code of the concept in the correct definition, comparing different definitions of the same concept; conceptualizing the objects, giving one's own examples to illustrate the concept, constructing counterexamples; drawing conclusions from the fact that of belonging the object to the given concept — naming the essential properties of the concept and assigning the same properties to the object that refers to the concept; including the concept into the generic and specific relationships with other concepts.

2.2. Teaching Proofs of Mathematical Statements in the In-Depth Learning of Mathematics

Irina Akulenko, Yuriy Leshchenko, Irina Vasylenko

Introduction

Students' mastering of the arts of proof, argumentation and disproof is one of the most important educational results in secondary education. A powerful potential in this context belongs to the teaching of mathematics and teaching proofs of mathematical statements in particular. Proof learning plays special importance for those students who study mathematics in depth. One of the weighty results of mathematics training is the students' ability not only to reproduce the proof of mathematical facts that are offered by the teacher or by the textbook, but also to design a personal proof or to disprove a proposed one. Manipulations should be based on the methods of scientific knowledge and methods of heuristic and logical thinking.

Background

General methodical aspects of teaching proofs of mathematical statements were considered in (Bradis[1], 1967; Grudionov[2], 1981; Dalinger[3], 2006; Metelsky[4], 1982; Slepkan[5], 2004;

[1] Bradys, V. M., Minkowski, V. L., & Kharcheva, A. K. (1967). *Errors in Mathematical Reasoning*: Manual for Teachers. Moscow: Prosveshchenie (in Rus.).

[2] Grudenov, Ya. I. (1981). *Learning Definitions, Axioms, Theorems*: Manual for Teachers. Moscow, Russia: Prosveshchenie (in Rus.).

[3] Dalinger, V. A. (2006). *A method of teaching students proofs of mathematical proposals*. Moscow, Russia: Prosveshchenie (in Rus.).

[4] Metelsky, N. V. (1982). *Didactics of Mathematics: General Methodology and its Problems*. Minsk, Belarussia: BGU (in Rus.).

[5] Slepkan, Z. I. (2004). *Psychological, pedagogical and methodical foundations of the developmental mathematical education*. Ternopil, Ukraine: Pidruchnyky i posibnyky (in Ukr.).

Tarasenkova[6], 2004; Poya[7], 1981; Franklin & Daoud[8], 2011; Ball, Hoyles, Jahnke, & Movshovitz-Hadar[9], 2002) and others.

The problem of teaching proofs of mathematical statements was studied by scientists in the following lines of investigation: the methods of designing and teaching proofs in elementary classes (Ball, & Bass, 2000[10], 2003[11]) in the algebra course[12] (Healy, & Hoyles, 2000), the methods of designing and teaching proofs in the course of geometry at secondary and high school[13] (Tarasenkova, 2004), the psychological and pedagogical bases of teaching students proofs[14] (Slepkan, 2004), the application of heuristics in search of the mathematical proof method[15] (Scafa, 2004),

[6] Tarasenkova, N. A. (2004). *The theoretic-methodical principles using of the sign-symbolic means in teaching mathematics of the basic school students*. (Doctoral disssertation). National Pedagogical Dragomanov University, Kyiv, Ukraine (in Ukr.).

[7] Polya, G. (1981). *Mathematical discovery: On understanding, learning and teaching problem solving*. New York: John Wiley & Sons.

[8] Franklin, J., & Daoud, A. (2011). *A Proof in Mathematics: an Introduction*. Kew Books.

[9] Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The Teaching of Proof, *ICM 2002*, Vol. III, 907-920. Retrieved from http://www.mathunion.org/ICM/ICM2002.3/Main/icm2002.3.0907.0920.ocr.pdf

[10] Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Eds.), *Yearbook of the National Society for the Study of Education, Constructivism in Education*,(p.p.193-224). Chicago: University of Chicago Press.

[11] Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.) *A Research Companion to principles and standards for school mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics. Retrieved from http://www.nctm.org/Handlers/AttachmentHandler.ashx?attachmentID=VJwObEnErFo%3d

[12] Healy, L., & Hoyles, C. (2000). From explaining to proving: a study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31, 396-128.

[13] Tarasenkova, N. A. (2004). *The theoretic-methodical principles using of the sign-symbolic means in teaching mathematics of the basic school students*. (Doctoral disssertation). National Pedagogical Dragomanov University, Kyiv, Ukraine (in Ukr.).

[14] Slepkan, Z. I. (2004). *Psychological, pedagogical and methodical foundations of the developmental mathematical education*. Ternopil, Ukraine: Pidruchnyky i posibnyky (in Ukr.).

[15] Skafa, E. I. (2004). *Theoretical and methodical bases of formation of methods of heuristic activity at studying mathematics in the conditions of introduction of modern technologies of training*. (Doctoral dissertation). Vasyl’

the formation and development of logical skills in teaching students math in depth[16] (Tarasenkova & Akulenko, 2013), the development of senior pupils' skills to prove mathematical statements in the process of learning algebra and the principles of analysis[17] (Kugai, 2007), teaching proofs in the in-depth learning of stereometry[18] (Yatsenko, 1999), the formation of students' skills of proving mathematical statements when learning functions in depth[19] (Kirman, 2010), the teaching of the elements of mathematical logic and the theoretical foundations of proof of math statements[20] (Akulenko, & Leshchenko, 2011), and others. Modern researchers focus their attention on benefits and warnings regarding the usage of computers in learning proofs, based on computer experiments in particular[21] (Shirikova, 2014).

Despite the wide range of pedagogical, psychological, and methodological studies, the problem of teaching the methods of proof and methods of searching a proof of mathematical statements to students who are learning mathematics in-depth remains relevant in school practice.

A teacher is to make a didactically balanced combination of a coherent argument and the accessibility of the method of proof, the heuristic and logical components of the process

Stus Donetsk National University, Donetsk, Ukraine (in Rus.).

[16] Tarasenkova, N. A., & Akulenko, I. A. (2013). The Problem of Forming and Developing Students' Logical Thinking in the Context of Subject Specialization in Secondary School. *American Journal of Educational Research*, 2 (12B), 33-40. Retrieved from http://pubs.sciepub.com/education/2/12B/7

[17] Kugai, N. V. (2007). *Forming the senior pupils' demonstration skills in the process of algebra and beginning of analysis learning*. (PhD thesis). National Pedagogical Dragomanov University, Kyiv (in Ukr.).

[18] Yatsenko, S.Y. (1999). *Education and instruction management for profound math course at basic school*. (PhD thesis). National Pedagogical Dragomanov University, Kyiv, Ukraine (in Ukr.).

[19] Kirman, V. K. (2010). *The system of methods of studying functions in school forms of mathematical profile*. (PhD thesis). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine (in Ukr.).

[20] Akulenko, I. A., & Leshchenko, Yu. Yu. (2011). *Elements of mathematical logic in the math course (plans of lessons)*. Cherkasy: Bohdan Khmelnytsky National University of Cherkasy, Ukraine (in Ukr.).

[21] Shirikova, T. S. (2014). *Methodology for teaching students of the basic school proofs of theorems in geometry using Geogebra*. (PhD dissertation). Northern (Arctic) Federal University, Arkhangelsk, Russia (in Rus.).

of finding a proof for students; to adjust the level of requirements for the proof of mathematical statements to the students' abilities and their level of cognitive interest. Such process should be conducted both at the lessons of mathematics and when receiving extra tuition.

We'll try to reveal the peculiarities of the teaching of proofs of mathematical statements in the elective course "Fundamentals of Cryptology" (for the students who learn mathematics in-depth).

Theoretical framework

The theoretical basis of the elective course "Fundamentals of Cryptology" is formed by the elements of the theory of information security, the theory of divisibility and modular arithmetic in the ring of integers (for the 8th grades students who study mathematics in-depth). It is designed for students of the 9th grade with in-depth learning of mathematics or for students of the 10th grade who study mathematics (informatics) at the profile level.

Caesar and Vigenere ciphers, the autokey cipher, asymmetric ciphers are of a particular importance in this course, since their acquisition is based on the basic concepts of number theory and the ability to perform adding and multiplying in a residue ring modulo n, using the properties of congruencies familiar to the 8th grade students.

The basic relation is the congruence relation. The basic concepts are: congruence, numbers that are congruent modulo, prime (composite) numbers, the greatest common divisor (GCD), the lowest common multiple (LCM), relatively prime integers, and canonical decomposition of a natural number. Basic mathematical facts are: the Euclidean division theorem, meaning of "a is congruent to b modulo m", the properties of divisibility among integers, the properties of congruencies, the properties of the GCD and LCM of two natural numbers, the properties of prime numbers, the Fermat's little theorem and its corollary, the theorem that substantiates the Extended Euclidean algorithm. The basic methods of activity are: the Euclidean algorithm, the application of properties of congruencies to perform their elementary transformations; the establishment of the fact that two natural numbers are relatively prime integers, the algorithm for finding the GCD of two natural numbers.

New concepts are: a complete residue system modulo *n* and reduced residue system modulo *n*, a linear congruence, a solution of the linear congruence, equivalent linear congruencies, elementary transformations of congruencies, inverse of *a* modulo *m*, linear representation of GCD of two natural numbers, a system of congruencies, a solution of a system of congruencies, Euler's totient function, a multiplicative function, a quadratic congruence, a quadratic residue modulo *n* and a quadratic nonresidue modulo *n*, a square root modulo a composite (or a prime) number. New mathematical facts are: necessary and sufficient conditions of the relative simplicity of two numbers, the property of the multiplicativity of Euler's totient function, the formula for Euler's totient function for an arbitrary natural number (or a prime power), Euler's theorem, the theorem on the number of solutions for the congruence $x^2 \equiv k \ (mod\ p)$, where k is the quadratic residue modulo a prime p, $GCD(k; p) = 1$, $p > 2$, the theorem on the number of the quadratic residues and nonresidues in the complete residue system, the Euler's criterion for determining whether an integer is a quadratic residue modulo a prime p, Chinese remainder theorem. New ways of activity: finding inverse of *a* modulo *m*, solving linear congruencies, solving systems of linear congruencies with two variables, recognition of solutions and solving the simplest quadratic congruencies by completing the square, the reducing of the congruence $ax^2 + bx + c \equiv 0 \ (mod\ m)$, where $(a; m) = 1$, to the binomial, solving quadratic congruencies modulo a natural *n* using Chinese remainder theorem; finding the square root modulo a composite (or a prime) number.

Methodology, results and discussion

We offer to prove the above-mentioned new mathematical facts at the level of logical rigor available to the students of this age category. The constructive proofs of statements will deserve more ample treatment.

As we know, a constructive proof in mathematics is a method of proof, which confirms the existence of a certain mathematical object by constructing a way of reproducing this object. It is opposed to a nonconstructive proof (also known as proof of existence theorem, or “pure” existence theorem), which proves the existence of a particular object without adducing of instances. We talk more about a constructive way of finding a proof, when the auxiliary constructions allow for giving a way to reproduce a particular object, and then for drawing conclusions that prove a mathematical statement as to the properties of a mathematical object.

We offer to use a constructive method for proving the multiplicative property of Euler’s totient function, the formula for Euler’s totient function for an arbitrary natural number (or a prime power), theorem on the existence of an inverse class of residues modulo a natural *n*.

But in our opinion, it is advisable to use both teaching to previously performed proof and to an independent finding of proof solutions by students. In theoretical investigations and in practice, the most widely used methods of teaching students of proofs are:

1) analysis and study of the solved proofs, demonstrated by the teacher or described in the textbook, with a view to their subsequent reproduction; independent finding of proof solutions by students by analogy with the given proofs; independent finding of proof solutions by students on the basis of the above-mentioned method; independent search and conducting of proofs (Slepkan[22], 2004; Dalinger[23], 2006 and others);

2) analysis of the solved proof and its presentation; unassisted discovery of facts, finding the solution of individual proof; the refutation of the proposed proof (Sarantsev[24],

[22] Slepkan, Z. I. (2004). *Psychological, pedagogical and methodical foundations of the developmental mathematical education*. Ternopil, Ukraine: Pidruchnyky i posibnyky (in Ukr.).

[23] Dalinger, V. A. (2006). *A method of teaching students proofs of mathematical proposals*. Moscow, Russia: Prosveshchenie (in Rus.)

[24] Sarantsev, G. I. (2006). *Teaching mathematical proofs and disproofs in school*. Moscow, Russia: VLADOS (in Rus.).

2006; Kugai[25], 2007 and others); 3) analysis and study of solved proofs; identifying the logical fundamentals of proofs and presenting them to students; unassisted finding of proof solutions by students' by analogy or with the teacher's help; unassisted finding of proof solutions on the basis of the knowledge of the logical basis of proof [26] (Stolyar, 1965). It is advisable to modify these methods when teaching students the constructive method of proof, taking into account the variability of additional auxiliary mathematical constructions (numerical sequences, expressions, functions, equations, etc.), their symbolic frames (tables, diagrams, graphs, etc.). We suggest relying on the first of the above-mentioned methods of teaching students the proof.

It is advantageous to familiarize the students with the reasoning method of a constructive proof of the mathematical fact on the example of proving the multiplicativity of Euler's totient function.

Let's consider the joint students' and teacher's activity gradually. *The motivation of studying the theorem is carried out by the teacher.* Euler's totient function $\varphi(n)$ is determined for all natural n and shows the number of nonnegative integers that are smaller than n and are relatively prime to n; thus $\varphi(1) = 1$.

For small values of natural n, the function $\varphi(n)$ can be found by simply counting the number of nonnegative integers smaller than and relatively prime to n: $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, $\varphi(8) = 4$, $\varphi(9) = 6$. But this way of finding becomes obviously too lengthy for large numbers. Therefore, it would be desirable to have a formula for finding the values of $\varphi(n)$. To find it, let us first fix on some properties of this function.

Theorem 1. *(This property is called the multiplicativity of Euler's totient function).*

[25] Kugai, N. V. (2007). *Forming the senior pupils' demonstration skills in the process of algebra and beginning of analysis learning*. (PhD thesis). National Pedagogical Dragomanov University, Kyiv, Ukraine (in Ukr.).

[26] Stolyar, A. A. (1965). *Logical problems of teaching mathematics*. Minsk, Belarussia: Vysshaya shkola (in Rus.).

For any relatively prime integers m and n the following equality holds: $\varphi(mn) = \varphi(m)\varphi(n)$. *The "discovery" of a mathematical fact by means of auxiliary constructions.* An example of constructing an auxiliary design and conducting further decisions is provided by a teacher. He involves students in discussion at definite stages of developing proof.

Let m and n be relatively prime integers. We should find how many numbers are smaller than the product mn and are relatively prime to it. To do this, we construct an auxiliary construction – a table containing m columns. We inscribe natural numbers from 1 to mn into it (Table 2.2.1).

1	2	3	...	m
$m + 1$	$m + 2$	$m + 3$	...	$2m$
$2m + 1$	$2m + 2$	$2m + 3$	...	$3m$
...	...	...	...	...
$(n - 1)m + 1$	$(n - 1)m + 2$	$(n - 1)m + 3$	...	nm

Table 2.2.1

The number a is the relatively prime to the product mn if and only if $GCD(a; m) = 1$ and $GCD(a; n) = 1$.

First, we find the numbers in the table that are relatively prime to m. In the first line, where all numbers are from 1 to m, the number of relatively prime integers to m is $\varphi(m)$. They form a reduced residue system modulo m (RRS). Let's mark them b_i. The numbers placed in each column belong to the same class of residuals modulo m. Therefore, if the number b_i is relatively prime to m, then any number in the column containing b_i is relatively prime to m too. Let's extract columns with numbers b_i, where $GCD(b_i; m) = 1$ (Table 2.2.2).

b_1	b_2	b_3	...	$b_{\varphi(m)}$
$m + b_1$	$m + b_2$	$m + b_3$	...	$m + b_{\varphi(m)}$
$2m + b_1$	$2m + b_2$	$2m + b_3$	...	$2m + b_{\varphi(m)}$
...	...	...	...	...
$(n - 1)m + b_1$	$(n - 1)m + b_2$	$(n - 1)m + b_3$	...	$(n - 1)m + b_{\varphi(m)}$

Table 2.2.2

Now we should find how many numbers in every column are relatively prime to n. Let us consider an arbitrary column (Table 2.2.3).

b_i
$m + b_i$
$2m + b_i$
...
$(n - 1)m + b_i$

Table 2.2.3

We have n distinct numbers. The general representation of these numbers is $mx + b$, where x consistently acquires values from 0 to $(n - 1)$, that is, x consistently acquires all values from the complete residue system modulo n (CRS). Let's prove that they all give different residuals modulo n, that is, the linear expression $mx + b$ consistently acquires all values from the CRS too. We use characteristics of the complete system (it must contain exactly n elements, and all of them must be pairwise non-congruent modulo n).

The given system consists of n numbers, since x acquires n different values in the expression $mx + b$. Let's prove that all these n obtained values are non-congruent modulo n with each other. Assume the opposite. Let $mx_1 + b \equiv \equiv mx_2 + b \ (mod\ n)$, when x_1, x_2 are not congruent modulo n. When simplifying the obtained congruence and taking into account that $(m, n) = 1$, we obtain $x_1 \equiv x_2 \pmod{n}$. We get the contradiction with the assumption. Consequently, each column is a complete residue system modulo n. So, each of them contains exactly $\varphi(n)$ numbers that are smaller than n and that are relatively prime to n. In general, the numbers that are relatively prime to the product mn in table 2 will be exactly $\varphi(n) \cdot \varphi(m)$. All of them are smaller than the product mn and are relatively prime to this product, that is, the value of Euler's function for the product mn is determined by equality $\varphi(mn) = \varphi(n) \cdot \varphi(m)$.

It's significant that during fixing a constructive method of proof it is important to carry out an extra work on the selection of the structure of proof, offering students to distinguish the main idea of proof, to analyze the auxiliary constructions (Tables 2.2.1–2.2.3); to construct a short plan of the proof, to fix arguments to the obtained facts. Equally important are students' constructions of proofs by analogy, by the specified structure, by the idea or method suggested by a teacher, using the same or a variant auxiliary construction.

For example, the consolidation of a constructive method of proof may be carried out in the course of the theorem proving – the value of Euler's function for the prime power, – having selected the structure of the proof and directing students to search for a way of proving in accordance to the system of corresponding questions (problems).

Theorem 2. *If p is a prime number and* $k \in N$, *then*:

$$\varphi\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right).$$

The proof is performed by students who give answers to the teacher's questions

1) Write all natural numbers from 1 to p^k in the form of a table containing p columns.

Expected students' response (Table 2.2.4)

1	2	3	...	p
$p + 1$	$p + 2$	$p + 3$	...	$2p$
$2p + 1$	$2p + 2$	$2p + 3$	...	$3p$
...	...	...	...	...
...	...	...	...	p^2
...	...	...	...	...
...	...	...	...	p^k

Table 2.2.4

2) Determine, what numbers in the table are relatively prime to p and how many of them are there.

Expected students' consideration. Numbers from the last column are multipliers of p, so they are not relatively prime to p. All other numbers of the table are not divisible by p, therefore, they are relatively prime to it.

3) Draw a conclusion on the quantity of numbers that are relatively prime to the number p^k.

Expected students' consideration. The numbers from the last column are not relatively prime to p, therefore, they are not relatively prime to p^k. All other numbers of the table are relatively prime to p, so they are relatively prime integers to p^k. Since the quantity of numbers in the table are p^k, and in the last column their number is p^{k-1}, the quantity of numbers that don't exceed p^k and are relatively prime to it are $p^k - p^{k-1}$.

4) Make the conclusion on the value of Euler's function $\varphi(p^k)$.

Expected students' consideration:

$$\varphi\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right).$$

We offer to master the theorem content and memorize the theorem formulation in the course of exercises.

Exercise 1. What is Euler's totient function value $\varphi(p)$, if p is a prime integer?
Expected students' consideration. If p is a prime integer, then the following equation can be written according to the Theorem 2:

$$\varphi\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right).$$

Corollary 1. *(is formulated by the students) If p is a prime integer, then $\varphi(p) = p - 1$.*

Exercise 2. How to determine the value of Euler's totient function for a natural number $n > 1$, which is given in the canonical form $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}$?

Expected students' consideration. Since Euler's function is multiplicative, so the following equality is performed according to the previous theorem:

$$\varphi(n)=\varphi(p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots\cdot p_s^{k_s})=\varphi(p_1^{k_1})\cdot\varphi\left(p_2^{k_2}\right)\cdot\ldots\cdot\varphi\left(p_s^{k_s}\right)=$$
$$=p_1^{k_1}\left(1-\frac{1}{p_1}\right)\cdot p_2^{k_2}\left(1-\frac{1}{p_2}\right)\cdot\ldots\cdot p_s^{k_s}\left(1-\frac{1}{p_s}\right)=$$
$$=n\left(1-\frac{1}{p_1}\right)\cdot\left(1-\frac{1}{p_2}\right)\cdot\ldots\cdot\left(1-\frac{1}{p_s}\right).$$

Corollary 2. *(is formulated by students) If a natural number $n > 1$ is given in the canonical form $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots\cdot p_s^{k_s}$, then:*

$$\varphi(n)=n\left(1-\frac{1}{p_1}\right)\cdot\left(1-\frac{1}{p_2}\right)\cdot\ldots\cdot\left(1-\frac{1}{p_s}\right).$$

As we can see, the auxiliary constructions can be presented by students unassisted (Tables 2.2.1 – 2.2.4) in different symbolic frames. It should be emphasized that auxiliary constructions can be various mathematical objects, such as: formulas, sequence of numbers, etc. (for example, in the proof of Euler's theorem).

Euler's Theorem. If $GCD(a;\ m) = 1$, then $a^{\varphi(m)} \equiv 1(mod\ m)$.

Prove. Consider the auxiliary construction: a sequence of $\varphi(m)$ arbitrary numbers x_1, x_2, ..., $x_{\varphi(m)}$, each of which is relatively prime to number m and these numbers are pairwise non-congruent to each other (it is a reduced residue system modulo m (RRS modulo m)).

We offer students to create and consider another auxiliary structure: a sequence of numbers ax_1, ax_2, ..., $ax_{\varphi(m)}$. To do this, they should multiply each number from a reduced residue system by a ($GCD(a;\ m) = 1$). We direct students' attention on studying the properties of numbers in this sequence: ax_1, ax_2, ..., $ax_{\varphi(m)}$.

Expected students' consideration.

1. There are $\varphi(m)$ numbers in the sequence.
2. Since all x_i are relatively prime to m, and a is a relatively prime to m, therefore, all ax_i are relatively prime to m.
3. Let's prove by contradiction the fact that they are pairwise non-congruent with each other. Assume that there is a pair of numbers such that $ax_i \equiv ax_j\ (mod\ m)$.

Since $GCD(a; m) = 1$, so both parts of the congruence may be divided by a, the result is $x_i \equiv x_j$ $(mod\ m)$, which contradicts the assumption.

Therefore, the sequence of numbers $x_1, x_2, ..., x_{\varphi(m)}$ has all three characteristic properties of RRS modulo m so it forms it. That is:

$$ax_1 \equiv x_{i_1} (\bmod m)$$
$$ax_2 \equiv x_{i_2} (\bmod m)$$
$$...$$
$$ax_{\varphi(m)} \equiv x_{i_{\varphi(m)}} (\bmod m)$$

The numbers $x_{i_2}, x_{i_2}, ..., x_{i_{\varphi(m)}}$ are the same numbers $x_1, x_2, ...,$ $x_{\varphi(m)}$ but, they are probably in a different order. Let's create one more auxiliary construction – we multiply the obtained congruencies. We find that:

$$a^{\varphi(m)}(x_1 x_2 ... x_{\varphi(m)}) \equiv (x_1 x_2 ... x_{\varphi(m)})(mod\ m).$$

Each of the numbers x_i is relatively prime to m, so the product $x_1, x_2, ..., x_{\varphi(m)}$ is relatively prime to m. So, after reduction we have:

$$a^{\varphi(m)} \equiv 1(mod\ m).$$

It is advisable to draw the students' attention to the fact that Fermat's little theorem directly follows Euler's theorem. Students studied the theorem in the 8th grade, however, the method of proving it was different.

Fermat's little theorem. If the natural number a is not divisible by a prime number p, then:

$$a^{p-1} \equiv 1(mod).$$

We suggest students to prove it independently, based on Euler's theorem (it is possible to do it orally).

The development of students' skills to prove statements within mathematical studies at the elective courses should be by proving both theoretical statements and solving problems. Here is an example.

Exercise 3. Prove that for natural numbers $n \geq 3$ the value of function $\varphi(n)$ is an even number.

Exercise 4. Using Euler's theorem, prove that the congruencies are hold:

a) $3^4 \equiv 1(mod\ 10)$; *b*) $7^{400} \equiv 1(mod\ 1000)$;
c) $9^{41} \equiv 9(mod\ 100)$; *d*) $11^{102} \equiv 121(mod\ 125)$;
e) $13^{40} \equiv 1(mod\ 41)$.

Conclusion

One of the important results of mathematics education at advanced or profile levels is the ability of students not only to reproduce the performed proofs of mathematical facts, but to design their own, based on the methods of scientific knowledge and methods of heuristic and logical thinking.

In teaching students the constructive method of proof it is necessary to focus students' attention on:

1) the individual or suggested by the teacher "discovery" of a mathematical fact by means of auxiliary constructions;
2) the consolidation of a constructive method of proof by distinguishing the structure, the basic idea of proof, auxiliary construction, constructing a short scheme of a given proof and making references to known facts in proving process;
3) students' establishing of proofs by analogy, according to the specified structure, or on the basis of the teacher's previously prescribed method or idea, using the same or a variant auxiliary construction; application of proved facts in solving proof exercises.

In this case, it is necessary to note the variability of auxiliary constructions and their symbolic frames, variations of methods of proof are possible.

2.3. The Efficiency of the Techniques for Enhancing Student Research Skills in Math Class: Pilot Testing

Olga Chashechnikova, Zoriana Chukhrai

Introduction

Growing demands on professional activity and the need to respond promptly to the situations not specified by the instructions require an ability to operate in a non-standard environment. The development of algorithmic thinking is problematic to the majority of young learners. As a result, the need to work in non-standard conditions, avoiding excessive algorithmization, causes difficulties[1] (Chashechnikova, 2011). Mathematical education becomes a strategic resource for the development of civilization, if it is directed not only at the formation of intellectual and professional-oriented knowledge and skills of the learners, but also at the development of their creative skills.

Review of the recent publications on the topic

Many psychologists, educators, and mathematicians in their research papers have focused on the development of creativity and thinking in general and mathematical education in particular. Scientists convincingly argue that the development of creative thinking takes place within various activities, including research ones.

In our opinion, the content of the concept of "creative activity" is most fully disclosed by V. Andreev[2] (1988): ***creativity*** is one of the types of human activity aimed at

[1] Chashechnikova, O. S. (2011). *Theoretical and methodological foundations of formation a+nd development of creative thinking of students in conditions of differentiated learning of Mathematics*. (Unpublished Doctoral dissertation). Sumy State Pedagogical University, Kyiv, Ukraine. (in Ukr.).

[2] Andreyev, V. I. (1988). *Dialectics of education and self-education of creative personality*. Kazan, Russia: Izd-vo Kazanskogo un-ta. (in Rus.).

solving a creative problem, which requires objective and subjective personal conditions and results in novelty and originality, is of personal and social significance. Creativity is impossible without the subject's thirst of cognition and sophisticated inquisitiveness. *Inquisitiveness* is a psychic feature, the property of human intelligence[3] (Goncharenko, 1977). Inquisitiveness manifests itself in the quest for comprehensive knowledge about the outer world and oneself, but develops in the process of cognition and practical development of a person in the objective reality. Proceeding from the fact that the study of the world is managed by the human activity called *research*, in some circumstances *learning activities under certain conditions can also be called research activity.*

We adhere to the point of view that *creative activity involves research, and research provides opportunities for creativity*[4] (Bilous, 2005). We rely on the definition of ***creative thinking*** as "the highest form of independent thinking of a person in the process of solving non-standard tasks, unconventional solving of traditional tasks; creating in the process of thinking new instruments, methods and techniques, the system of which can be used in the future to fulfill a wider range of tasks"[5] (Chashechnikova, 2006). The development of creative thinking of an individual takes place in the process of various activities in general and ***research activities***, in particular, which O. Savenkov[6] (2006) defines as "*a special kind of intellectual and creative activity* that is generated as a result of the functioning of the mechanisms of search activity and is constructed on the basis of research behavior". Therefore, we consider research

[3] Goncharenko, S. U. (1997). *Ukrainian Pedagogical Dictionary.* Kyiv, Ukraine : Lybid. (in Ukr.).

[4] Bilous, S. (Yu). (2005). *Development of research abilities of senior pupils in the process of activity of the Small Academy of Sciences (on the material of Physics).* (Unpublished Doctoral dissertation). National Pedagogical University MP Dragomanova, Kyiv, Ukraine. (in Ukr.).

[5] Chashechnikova, O. S., & Karlash, O. V. (2006) Enhancing the efficiency of developing the liberal arts' high school students' creativity in Math class... *Pedahohichni nauky*, *1*, 219-228. (in Ukr.).

[6] Savenkov, A. I. (2006). *Psychological Fundamentals of the Research Approach to Learning*. Moscow, Russia: Os'-89. (in Rus.).

abilities as a component of creative thinking.The concept of "research abilities" we define as the *personality properties that allow to analyze and critically evaluate the phenomena (objects, events), to reveal their characteristic features and interrelations, to predict the possible effects of such changes on the phenomenon* ***(object, event)***[7] (Chukhrai, 2013).

Due to the fact that in the process of teaching mathematics to the students of non-mathematical specialties, the training of mathematicians is not a goal, and taking into account different levels of students' training, we distinguish educational abilities (the ability to learn mathematics) and creative mathematical abilities (related to the independent creation of the original).

Classical is *the definition of the abilities to study mathematics,* offered by B. Krutetskyi[8] (1968): "individual psychological peculiarities <...> of mental activity corresponding to the requirements of educational mathematical activity and stipulate, taking into account other equal conditions, the success of the creative mastery of mathematics as an educational discipline, and relatively quicker, easier, and deeper mastering of knowledge, skills and abilities in the field of Maths".

In previous scientific studies[9] (Chashechnikova, 1997) mathematical abilities are considered as the ability to solve creative mathematical problems, as "individual psychological features of a person, contribute to higher productivity of his mathematical activity, allow to use non-standard ways and methods during this activity, creating a comparatively new result (or qualitatively new) product of mental mathematical activity".

The basic and expanded systems of components of mathematical abilities are offered.

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[7] Chukhrai, Z. B. (2013). *Development of research abilities of students of economic specialties within teaching mathematics.* (Unpublished Doctoral dissertation). Sumy State Pedagogical University, Sumy, Ukraine. (in Ukr.).

[8] Krutetskyi, V. A. (1968). *Psychology of school children's mathematical abilities*. Moscow, Russia : Prosveshchenye. (in Rus.).

[9] Chashechnikova, O. S. (1997). *Development of Mathematical Abilities of Primary School Students.* (Unpublished Doctoral dissertation). The Institute of the Pedagogical Sciences of Ukraine, Kyiv, Ukraine, Kyiv, Ukraine. (in Ukr.).

Considering research abilities as one of the components of creative thinking, we offer the characteristics of creative thinking[10] (Chukhrai, 2013), to manifestation which we can trace the dynamics of the research abilities of college students in the field of economics in the process of teaching mathematics: non-consistency of thinking (NCTh), critical thinking (CrT), independence of thinking (ITh) and the ability to self-organization (ASO), multiplicity of thinking (MTh), predictability of thinking (PrTh).

We have created a methodological system for teaching mathematics which is focused on students' development.

The purpose of the article is to present the results of an experimental verification of the effectiveness of the implementation of the developed methodical system of mathematics training aimed at developing research abilities, under higher mathematics education at economic colleges.

Presenting the essential material

The majority of college students of economic specialties (according to the results of our questionnaire[10]:Chukhrai, 2013) were graduates of non-mathematical classes; their average grade in maths is 7 (based on a 12-point rating system). At the maths classes in colleges the use of reproductive teaching methods, explanatory and illustrative method are dominant; assigments of reproductive and reconstructive nature are often offered.

A detailed analysis of the results of our test control of students' knowledge and skills in mathematics suggests that students' difficulties may arise even in the process of solving standard assigments; some students ignore open-form assigments (especially those that require a detailed answer): only 11.59% of the respondents prefer assigments with an unrnown algorithm. Students who have been tested for input, actively use new information and communication technologies in the learning process, but only 31.79% of them work

[10] Chukhrai, Z. B. (2013). *Development of research abilities of students of economic specialties in the process of teaching mathematics.* (Unpublished Doctoral dissertation). Sumy State Pedagogical University, Sumy, Ukraine. (in Ukr.).

unassisted with textbooks and manuals on maths. The results of the questionnaire show: non-trivial ideas for solving the assignment "often" appear in 10.93% of respondents, "rarely appear" in 49.01% of respondents, although it delivers them satisfaction. 38.41% of the interviewed students prefer to work on reproductive problems; 37.09% – on reconstructive ones; only 22.52% – on variational and even creative ones.

So, it is necessary to stimulate students unaided educational and cognitive activity, in particular – that of research character.

According to the results of the survey college students underestimate the importance of applying knowledge and skills in mathematics in professional activities, the role of mathematics in the development of research abilities. Therefore, we consider it expedient to explain to the students the role of advanced research abilities in their future professional activities.

Unconventional thinking allows to apply or use different ways of solving professional problems. The professional should critically evaluate the information provided to him, determine the effectiveness of the activity. So, ***critical thinking*** is important. ***Independence of thinking*** contributes to the identification of initiative in the course of professional activity, and ***prognostic thinking*** – to evaluate the options for the development of the situation, to find out the hypotheses, to define goals, to develop a focused scheme of action. ***Multiplicity of thinking*** gives the opportunity to use various information, to analyze and interpret it (in particular, to interprete it into the language of Mathematics); to solve professional problems by mathematical methods, substantiating the expediency of their own actions; to use the results of previous researches when performing new tasks. Ability of self-organization is necessary in any professional activity. It provides an opportunity to navigate in a new unfamiliar situation, to reach the goal, without paying attention to emerging difficulties, to independently manage their own activities.

We have offered an approach of creating a system for teaching mathematics to the students of economic colleges, which we have generalized and used in teaching mathematics to the students of other specialties. The use of this system makes it possible to make mathematical knowledge an effective instrument of solving real professional problems; the existing level of the development of research abilities serves as a pillar of their further improvement. The effectiveness of this approach is substantiated.

According to this approach, the tasks of "conditional research" are used (the concepts "conditional-creative", "conditional-research" were introduced by us in[11] (Chashechnikova, 2011), and are supplemented by the tasks of the applied orientation.

A comparative analysis of the application of the forms, methods and techniques of teaching Maths was carried out and it was found out that it is expedient:

- to strengthen the professional direction of the content of lectures in higher Maths;
- to involve students to find the possibilities of using the mathematical apparatus to solve industrial problems;
- to offer non-standard assigments, complex assigments;
- to increase the volume of assigments of a problematic nature, assigments for research (or with elements of research) during the study of all topics of the course;
- apply business games simulating the real production situation;
- to emphasize the students' attention to independent activities (in particular, the creation of reference notes).

The traditional goals and tasks of teaching higher Maths in economic colleges (italics) are supplemented and refined:

1) the application of mathematical knowledge in the process of solving economic assigments, the construction of economic and mathematical models, *the orientation of students' activities on the development of rational methods for their research on the basis of qualitative and quantitative analysis, the development of appropriate practical recommendations;*

[11] Chashechnikova, O. S. (2011). *Theoretical and methodological foundations of the formation and development of creative thinking of students under differentiated learning of Mathematics.* (Unpublished Doctoral dissertation). Sumy State Pedagogical University, Sumy, Ukraine, Kyiv, Ukraine. (in Ukr.).

2) the personal focus of teaching Maths (long-term home assigments and considering an individual time of solving, enabling students to work at a higher level of difficulty under the conditions of the unaided study of a significant amount of educational material);

3) development of *students'* analytical thinking and *research skills*.

Constant emphasis on the professional orientation of educational material in Maths contributes to the development of research abilities of students. We have developed an author's educational complex in Maths based on the specifics of the training of students in economic colleges. The complex contains the curriculum of the discipline "Higher Mathematics", a teaching and methodological manual on higher mathematics, didactic materials oriented at the development of research abilities of students of economic specialties, a computer program of advisory and educational character "Consultant-simulator", methodical recommendations for teachers on the development of research abilities within higher mathematics courses. The author's curriculum[12] (Chukhrai, 2009) describes the interdisciplinary connections of each topic of the higher mathematics course with the subjects of economic disciplines as to motivate the students to study (in particular, the study of the topic "Elements of the theory of matrices and determinants" is connected with the study of the themes in other disciplines like "Financial Accounting", "Statistics", "Economy of the Enterprise", and "Placement of Productive Forces").

Features of the proposed teaching aids[13] (Chukhrai, 2012): the theoretical material is presented in the form of "question research"; the appearance of the new formula is accompanied by an indication of a specific example of its application when solving a specific problem (in particular, professionally directed). A brief presentation of the corresponding professionally directed theoretical material

[12] Chukhrai, Z. B. (2009). *Higher Mathematics*. Malyn, Ukraine. (in Ukr.).

[13] Chukhrai, Z. B. (2012). *Higher Mathematics: theory, practice, and application in the professional activity of an economist*. Rivne, Ukraine : Volynski oberehy. (in Ukr.).

facilitates the student's motivation for active educational and cognitive activity and allows to consider in mathematics classes a sufficient number of non-standard assigments of economic content, to realize interdisciplinary connections. The questions for self-control are proposed, which facilitate the formation in students the methods of mental actions; a system of assigments has been developed, the complexity of which is gradually increasing, and some of them provide different ways of solving. All this motivates independent research activity of students. The system also contains assigments of economic content. It is taken into account that the selection of assigments of a professional character requires the accurate use of economic terminology, the nature of the assignments should reflect real economic (production) processes. The tasks of three levels of complexity for unaided solving (*the third level* – the research character, the scheme of their solution "is opened" by the student) is offered after studying a sufficiently large block of theoretical information. So, the student does not have the instruction on which training material is to be used, they are not limited in choosing a method for solving the problem (element of research). The author's computer program of advisory-educational character "Consultant-simulator"[14] (Chashechnikova & Chukhrai, 2010) (the methodology of work with it is described in[15]: Chukhrai, 2013) assists in rational organization of the unaided educational and cognitive activity of students, "mergence" of time for the implementation of research activities at higher mathematics classes. The program is adapted to the individual pace of students' learning and takes into account their level of preparation, interests and abilities, which helps to prevent the overload of students and teachers of Maths. The program implements *advisory and training functions*; provides the *student* with the *opportunity* to work out the skills of

[14] Chashechnikova, O. S., & Chukhrai, Z. B. (2010). A.s. *35866 Ukraine Computer program «Consultant-simulator»* : № 35866 vid 29.11.2010, № 36047; zaiavl. 29.09.2010. (in Ukr.).

[15] Chukhrai, Z. B. (2013). *Development of the research abilities of students of economic specialties within teaching mathematics.* (Unpublished Doctoral dissertation). Sumy State Pedagogical University, Sumy, Ukraine. (in Ukr.).

prompt execution of simple tasks that are part of research tasks; to trace the dynamics of their own growth; provides an opportunity for a two-way communication "teacher ⇔ student". *A student* has a motivated need of self-education; develops the ability to concentrate, act purposefully.

In choosing the goal for developing student's research abilities, a Maths teacher should be familiar with the specifics of research activities, be able to apply mathematical apparatus for the study of mathematical models of economic situations; use forms, methods and techniques that encourage students to receive cognitive satisfaction in learning Maths. The instructor should involve students in research, organize and manage it, motivate self-education; to encourage students to original ideas, to provide them with the opportunity to independently verify the correctness (falsity) of their reasoning, in accordance with the solution obtained in the requirements formulated in the task. Taking into account the results of O. Chachechnikova's research, the cooperation of a Maths teacher (as well as teachers of professionally related disciplines where the mathematical apparatus is used) and students, and the creation of favorable conditions for the transition of cooperation to the level of interaction, are important for the development of the students' research abilities: the psychological atmosphere of equality and tolerance, partnership in learning and educational activities; providing the students with the opportunity to express their thoughts on solving assigments of a research nature, motivation for their analysis, justification of expediency; the presence of students in the course of study time for the independent construction of the hypothesis, its justification (or refutation), the motivation of students for hard work for self-improvement.

The proposed methodological system for developing the research abilities of students of economic colleges was tested experimentally during in 2005-2011academic years.

The pedagogical experiment was conducted on three stages on the basis of the institutions of the I-II accreditation levels of Rivne region: Bereznivskyi Forestry College, Western Ukrainian College "Polissia" (Berezno Rivne Region), Rivne

Cooperative Economics and Law College, Rivne College of Economics and Business, Rivne Economic-Humanitarian and Engineering College. Students of Lubenskyi Forestry College, Kremenets Forestry College, the V. Sulka Shatsk Foresty College, Malynsk Forestry College were approbated. The students of the institutes of the III-IV levels of accreditation, namely: the Ukrainian Academy of Banking of the NBU (Sumy) and the National University "Ostroh Academy" (Ostroh, Rivne region) were taken into the experiment.

All the participants of the experiment (973 students) were divided into three groups: control (CG), partially experimental group (PEG) and experimental group (EG) (Table 2.3.1).

Group	Number of participants of the experiment					
	2005/2006	2006/2007	2007/2008	2008/2009	2009/2010	In all
EG	63	69	66	64	60	322
PEG	65	70	64	62	62	323
CG	65	70	62	66	65	328
In all	193	209	192	192	187	973

Table 2.3.1 Distribution of experiment participants by years.

At the stage of the *confirmatory experiment*, the state of the problem under study in the theory and practice of learning was studied by analyzing sources, observing students in the process of learning and teachers in the process of teaching, consultations, analysis of their written tests and control works, etc. An anonymous questionnaire was conducted also.

Among the university students there were 45.45% of the graduates of the physics and mathematics classes, among the college students there were 5.56% of such kind of students; participants in the Olympiads in Maths there were 72.72% of university students and 13.73% – of college ones; level of knowledge of respondents from the course of school Maths: for colleges – 7 points out of 12, for universities – 11 points out of 12. Only 7.84% of college students and 22.73% of university students are ready to process the material on their own. The difference in outcomes among university students and college ones convinces: it is necessary to organize a college-centered learning process focused on developing students' research abilities through adapted approaches.

The following task was set: to develop a methodical system of studying Higher Maths in colleges with the focus on the development of research abilities of students.

Research experiment. A methodical system for developing research abilities of college students was created, tested, and corrected; an author's (authentic) curriculum of discipline[16] (Chukhrai, 2009), a teaching and methodical manual[17] (Chukhrai, 2012), an authentic computer program "Consultant-simulator"[18] (Chashechnikova, & Chukhrai, 2010) were developed; methodical recommendations for teachers on the appropriate selection of forms, methods and techniques for organizing training classes focused on the development of students' research abilities were created; recommendations on the application of information and communication technologies in the educational process; didactic materials were developed[19] (Chukhrai, 2011).

571 students participated in the *formative experiment* (4 academic years). For the division of students into groups the comparative test control of knowledge (CTCK) in Maths was conducted (maximum point is 64): the knowledge and skills of students from the school's mathematics course necessary for studying higher mathematics at the higher educational establishment were checked up. Three groups were formed:

- control group (CG, 193 participants), whose students studied according to the traditional system;
- partially experimental group (PEG, 188 participants), where some elements of the methodical system developed by us were used (in particular, the use of author's curriculum and computer program of counseling and teaching character);
- and experimental group (EG, 190 participants), which fully implemented the process of teaching higher mathematics

[16] Chukhrai, Z. B. (2009). *Higher Mathematics*. Malyn, Ukraine. (in Ukr.).

[17] Chukhrai, Z. B. (2012). *Higher Mathematics: theory, practice, and application in the professional activity of an economist*. Rivne, Ukraine : Volynski oberehy. (in Ukr.).

[18] Chashechnikova, O. S., & Chukhrai, Z. B. (2010). A.s. *35866 Ukraine Computer program «Consultant-simulator»* : № 35866 from 29.11.2010, № 36047; declared. 29.09.2010. (in Ukr.).

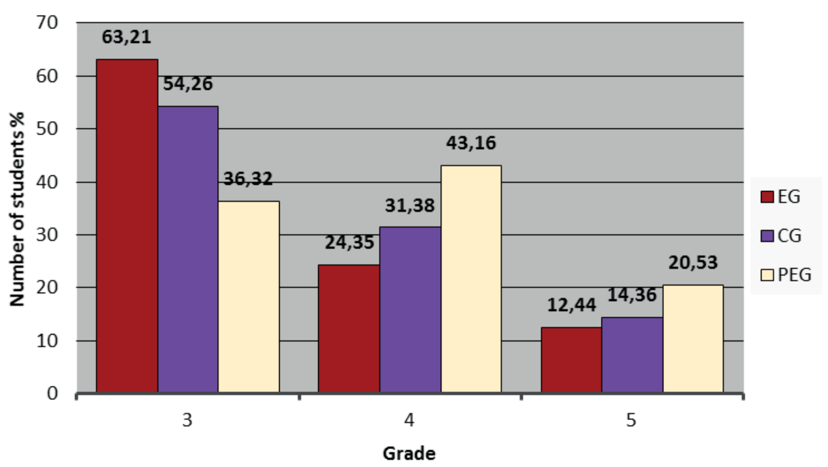
[19] Chukhrai, Z. B. (2011). *Studying Higher Mathematics Unaided*. Berezne, Ukraine : Bereznivskiy lisoviy koledg. (In Ukr.).

according to the methodical system developed by us. A check was made on the correct distribution of respondents by groups.

The groups were selected so that the selective average score for CTCK students of CG was not lower than that one of the students of PEG and EG (comparison of the average points of the groups was carried out according to the Student's criterion[20] (Bobyk, Berehova, & Kopytko, 2007). It was determined: the average values of the success in school course of Maths among students of the three groups at the beginning of the formative experiment did not differ significantly.

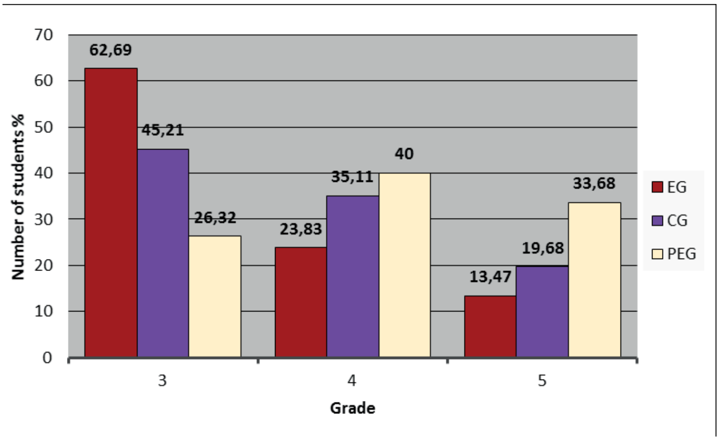
For diagnostics of system knowledge students of higher mathematics two test three-level modul control works (TT-MCW-1 and TTMCW-2) were offered for all of the respondents at the beginning and at the end of the experiment. All the tasks had elements of research. The results of TTMCW-1 and TTMCW-2 are shown in Figure 2.3.1 (a, b).

Figure 2.3.2 (a, b) illustrates the number of correctly completed assignments by the respondents of all three groups (in percentages). To compare the general averages, Student's criterion was used. The quantities necessary for its calculation are presented in Table 2.3.2).



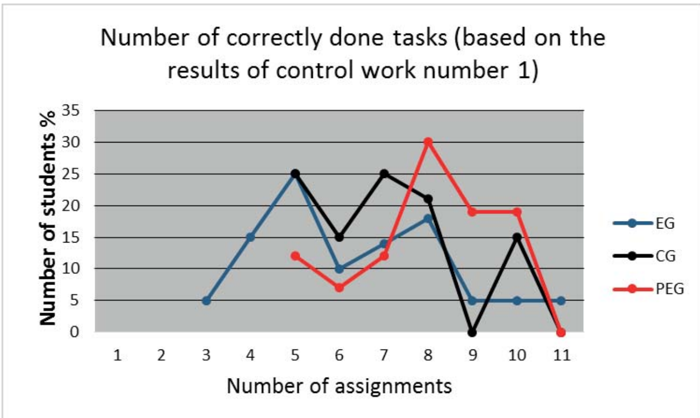
a

[20] Bobyk, O. I., Berehova H. I., & Kopytko B. I. (2007). *Probability Theory and Mathematical Statistics*. Kyiv, Ukraine: VD «Profesional». (in Ukr.).

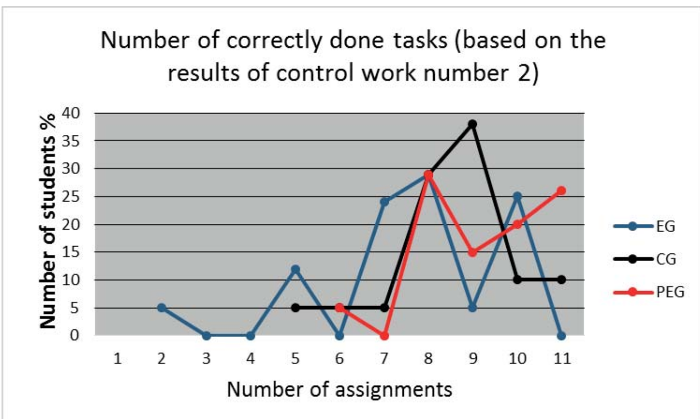


b

Figure 2.3.1 (a, b) Results of TTMCR-1 and TTMCR-2.



a



b

Figure 2.3.2 (a, b) The line of the number of completed assignments TTMCR-1 and TTMCR-2.

Number of students			Sampling averages	Sample variances	Sample mean square deviation	Corrected sample dispersions
1			2	3	4	5
EG	x	$n_x = 190$	$\overline{x_s} = 3{,}84$	$D_s(x) = 0{,}55$	$\sigma_s(x) = 0{,}74$	$S_x^2 = 0{,}56$
PEG	y	$n_y = 188$	$\overline{y_s} = 3{,}6$	$D_s(y) = 0{,}53$	$\sigma_s(y) = 0{,}73$	$S_y^2 = 0{,}53$
CG	z	$n_z = 193$	$\overline{z_s} = 3{,}49$	$D_s(z) = 0{,}5$	$\sigma_s(z) = 0{,}7$	$S_x^2 = 0{,}5$

Table 2.3.2 Calculations to use Student's criterionon.

The corrected dispersions of the three samples under investigation appeared to be different, therefore the equality of general dispersions was confirmed by the Fisher-Snedekor criterion.

To confirm that the average values of mathematical achievement in the investigated groups groups did not differ significantly at the beginning of the molding experiment, the zero and alternative hypotheses were formulated in pairs for groups:

1) for EG and PEG: zero $H_0 : M(x) = M(y)$ at the alternative $H_\alpha : M(x) > M(y)$;

2) for EG and CG: zero $H_0 : M(x) = M(z)$ at the alternative $H_\alpha : M(x) > M(z)$;

3) for PEG and CG: zero $H_0 : M(y) = M(z)$ at the alternative $H_\alpha : M(y) > M(z)$.

Zero hypotheses about the equality of general averages in groups will make it possible to verify whether the quality of the residual knowledge of the participants in the experiment depends on the use of the proposed methodological system. Calculated Observed Values of the Student's Criterion.

Taking into account the statistical processing of sample aggregates EG, PEG and CG, we can talk of the deviation of the null hypothesis for the cases of EG and PEG ($Z_1^* = 3{,}25 \notin (-\infty;\ 2{,}34]$) EG and CG ($Z_2^* = 4{,}73 \notin (-\infty;\ 2{,}34]$) and its acceptance for the general average groups чег and КГ ($Z_3^* = 1{,}5 \notin (-\infty;\ 2{,}34]$). The traditional technique does not provide the level of training that is achieved with the help of the proposed author's technique, the use of only certain elements of the developed methodological system does not contribute to improving the quality of knowledge of college students. In EG, the selective mean value of the results of TTMCK-1 is higher than in the CG and PEG.

To confirm the assumption of the dependence between the level of development of students' research abilities and their success in Higher Maths, a sample correlation coefficient for EG / PEG, EG / CG, and CG / CG was calculated (Table 2.3.3).

To check the hypothesis about the significance of the sample correlation coefficient, the observable value of the criterion is calculated using the formula: $T_{obser} = \frac{r_s\sqrt{m-2}}{\sqrt{1-r_s^2}}$, where m is the

Groups	Sample correlation coefficient (r_s)	Observed value of Student's criterion (T_{obser})
EG/PEG	0,62	10,83
EG/CG	0,44	6,7
PEG/CG	0,98	67,17

Table 2.3.3 Results of calculating the sample correlation coefficient.

number of students in the experimental group (or partially experimental for the PEG / CG variant), r_s – the sample correlation coefficient. Due to Table[21] (Zhluktenko, Nakonechnyi, & Savina, 2001), the critical points of Student's distribution tp (0.01; 190) = t_{cr} (0,01; 188) = 2,58.

The critical value t_{cr} does not exceed the empirical value T_{obser}, that's why there was reason to assert that the null hypothesis is rejected. Thus, the level of mastering of students' knowledge and skills in Higher Maths depends on the level of development of students' research abilities.

At the end of the study of the Higher Math, another modular control work was carried out, and the statistical distribution of the frequencies was constructed for processing the results obtained with TTMCW-2 (Table 2.3.4).

[21] Zhluktenko, V. I., Nakonechnyi, S. I., & Savina S. S. (2001). *Probability theory and mathematical statistics: in 2 ch.* Kyiv, Ukraine : KNEU (in Ukr.).

Group		Grade		
		„3”	„4”	„5”
EG	x	50	76	64
PEG	y	85	66	37
CG	z	121	46	26

Table 2.3.4 Results of TTMCW-2 presented by students of CG, PEG and EG.

Comparing the average sample, it was found that the ave-rage points obtained by students for TTMCW-1 and TTMCW-2 in EG, PEG and CG differ significantly (Table 2.3.5).

Group		MCW-1	MCW-2	Differences
EG	$\overline{x_s}$	3,84	4,07	0,23
PEG	$\overline{y_s}$	3,6	3,74	0,14
CG	$\overline{z_s}$	3,49	3,51	0,02

Table 2.3.5 Average points obtained by students for doing TTMCW-1 and TTKMCW-2

Based on the correlation analysis of the results of TTMCW-1 and TTMCW-2, it has been established that there is a strong positive correlation between the level of development of students' research abilities and the quality of residual knowledge and skills in Higher Maths; the correlation coefficient is significant ($r_s = 0{,}9999$).

On the basis of the conducted static and correlation analysis of the results of control works in Higher Maths we conclude on the effectiveness of the implementation of a methodical system developed by us for the improvement of the quality of students' training in the economic specialties of colleges in the educational process.

Conclusions

The experimental analysis of the main provisions of the study, the correction of the results confirmed the effectiveness of the implementation in the college's educational process of the proposed system of teaching Maths, with the aim of developing students' research abilities.

2.4. Intellectual Learning Tools and the Study of Functions in High School Mathematics Curriculum

Vadym Kirman

The most specific factor in the study of functions is the use of the intellectual learning tools. The intellectual learning tools include[1] (Tarasenkova, 2002):

1) the content of training as such;
2) sign and symbol means of fixing content and activities with it;
3) means of control and management of educational cognitive activity, which include questions, exercises, tasks;
4) a complex of acquired knowledge, skills and abilities of general and subject-oriented nature.

The contents of teaching the substance of functions covers the study of specific classes of elementary functions (unit of "Elementary functions"), the issue of studying the functions and foundations of mathematical analysis (unit "Mathematical analysis"), related issues of computing (unit "Calculation"), and mathematical simulation (unit "simulation"). The theoretical multiplicity of the concept of function (the unit set-theoretic approach to the study of functions) is the core of the content function line, which is formed both autonomously and parallel in the combinatorial (block "combinatorics") and geometric (block "geometric transformations") tasks. The model of the corresponding structure of the contents is shown in Figure 2.4.1

In connection with the analysis of the logical structure of the training content, we offer the use of another means of learning: *the graph of the logical connections in the system of statements*. A graph of logical relationships is an oriented graph, whose vertices are statements, and we interpret the edges as logical implications, that is, if the graph contains a chain G→P, it means that P is used in proof P. Similar

[1] Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics : Monograph*. Cherkasy, Ukraine: Vidlunnya-Plus (in Ukr.).

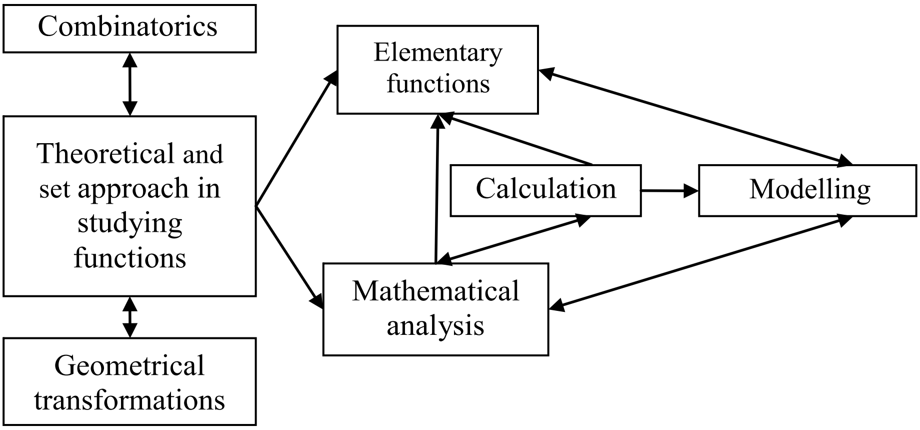


Figure 2.4.1 Model of the contents of teaching the content line of functions.

graphs can be used in defining complex concepts. For a strict theory, the graph of logical connections should not contain "closed circles," but the school course is based on the laws others than those of formal theory, therefore, such circles may appear. The study of how to get rid of the circle is in itself a rather difficult task for students. The chain of links, obviously, can be very long, so always when studying the topic there should be pointed out "questionable" statements and those that we can "safely" apply.

We use the corresponding graphs for new theorems. For example, within the profound study of mathematics, Lagrange's theorem becomes the root vertex of a tree representing Lagrange's theorem.

By the thesaurus we mean "verbalized entity of concepts about the world, including cognitive settings"[2] (Shreider, & Sharov, 2002). The same authors admit, that thesaurus determines the system of the semantic connections, and relations. From the positions of the systemic approach we have field-proved the notion of the functional thesaurus (the thesaurus of functions)[3] (Kirman, 2009). Actually, the functional thesaurus is the set of notional function spaces and elements of different sets, ranged by hierarchical levels. Thus, if on the first level the notional space of real numbers

[2] Shreider, Y. A., & Sharov, A. A. (1982). *Systems and models*. Moscow, Russia: Radio i svyaz'. (p. 118) (in Rus.)

[3] Kirman, V. K. (2009). Functions thesaurus as an intellectual learning tool. *Cherkasy University Visnyk. Series Pedagogical sciences, 162*, 77–83 (in Ukr.).

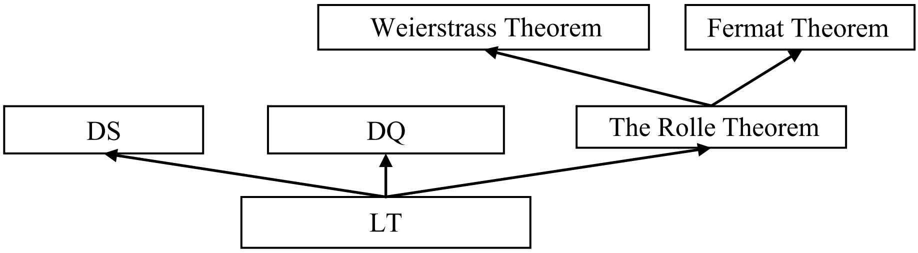


Figure 2.4.2. The logical connections tree for the Lagrange theorem.

is placed, on the second place there is that of numerical functions, the third level is the functionals and the operators (for example, the operator of finding the derivative, the functional of finding the definite integral), and the fourth level is the level of the mappings defined on the sets of operators and functionals. Studying different aspects of the contents line of functions is accompanied by the changes of the functional thesaurus, for example, by the appearances of new objects, or bifurcations (we look at an ordered pair of numbers, for example, as at a function, a problem on a set $\{1;2\}$, so this object is subdivided into two levels). The teacher by all means should take into consideration the state of the students' functions thesaurus. Thus, for a person, whose second level of the thesaurus has been filled only with direct and inverted proportionals, it would be natural to have such an erring (but widespread) train of thought: "As a result of the experiment, we observed that with an increase in the value of I, the value of U increases as well, from which we conclude that $U=rI$, where r is some number".

Sign and symbolic means of teaching are described in detail and classified in the studies of N. A. Tarasenkova. So, in the works of Tarasenkova[4] (2003) objective texts, terminology, symbolics, mathematical sentences, teaching texts, texts of problems, texts of questions, and pictography are referred to the verbal sign and symbolic means; the nonverbal sign and symbolic means are the images of geometric figures, contents and graphic interpretational, tables, diagrams, graphics, schemes, analytical configurations, illustrations,

[4] Tarasenkova, N. A. (2003). The theoretic-methodical principles of using the sign and symbolic means in teaching mathematics to the basic school students (Doctor's thesis). Cherkasy national university. Cherkasy. Ukraine (in Ukr.).

layouts, constructions, and plastics. We will consider the specifics of some sign and symbolic means separately, it being manifested in studying functions.

We mean, first of all, the functional symbolics. To it we refer designations of functional dependences (like, $y=y(x)$, $y=f(x)$), the signs of binary operations (functions of two variables), designations of functions, designations of geometric transformations, designations for boundaries of functions, derivatives (the example of an operator), undefined, defined integrals (the example of a functional), differentials, designations of operations with functions (the inverted function, compositions of functions) and others. A part of functional symbolics is connected with the theory and set symbolics: designations of mapping from one set to another ($g: X \to Y$), the area of definition and the set of meanings of the function ($D(f)$, $E(f)$). In learning the functional symbols one should pay attention to the following:

1. Linguistic origin of the symbol, especially in the cases, where symbols are shortenings of the corresponding words of the foreign origin.

2. Historical and mathematical aspect of the origin of the corresponding symbol. It is of importance, because sometimes at the stage of the initial acquaintance the introduction of the certain symbolics seems ungrounded. The symbolics reflects style of thought of those mathematicians who introduced it. Thus, for example, it is very hard to argue the symbols of the defined integral $\int_a^b f(x)dx$, the derivative $\frac{dy}{dx}$ without an infinitesimal context, in the case of the first symbol – as the sum of the infinitely large quantity of the infinitely little addends and in the case of the latter symbol – as the relation of the infinitely little gain of the function to the infinitely little gain of the argument.

3. The possibility to use the alternative symbolics.

4. The operational aspect, the convenience of the use of the symbolics, the comparison with the possibilities of the use of the alternative symbols.

An important tool is the introduction of new symbols. Functional symbolics offers a simple way of doing it. The most famous way is that of introducing the auxiliary functions while solving problems (a possible fragment of the mathematical text: "Suppose $v(x)=\ln|\sin x|+\sqrt{1-x^2}$"). Auxiliary functions can also occur in formulating problems. Introduction of the auxiliary functions with the help of traditional functional symbolics (the functions from several variables included), as a rule, is not challenging for students. Besides, the significant part of the 9-10[th] graders cannot cope, for example, with such task: "Calculate $7\circ 9$, if $a\circ b=\max(a;b)+b^{a-5}$". Here, the function from two variables is written as a binary operation unusual for the students. In mathematical practice very often there arises the necessity for transformation from one denomination system to another, that is why there is the need to develop in students the skills of working with mathematical texts with an alternative (including new) symbolics. We call such skills *the skills of the symbolic transposition, it being* analogical with how the musicians read the text while passing from one tonality to another. Constant introducing of new alternative symbols and the work with them is the best way to develop such skills. In fact, both – the commonly applied symbols as well as specific ones can be used. Inner (specific) symbols are used as auxiliary ones in the texts for proving the theorems and solving problems, and as special ones while formulating problems for the development of skills of symbolic transposition.

In the same way one can speak about the commonly used (nominee and auxiliary) and inner terminology. The occurrence of the inner symbolics and terminology can be connected not only with the corresponding problems, but with the traditions, formed in some environment and individual fantasy. New possibilities are opened by the use of the inner terminology and symbolics in the system of propaedeutics while working with the notions in the background regime.

While studying functions functional tables have the principle meaning. One can distinguish local and global functional tables. Local functional tables describe the meaning of functions for arguments, most often occurring in the training tasks. An example is the table of meanings of trigonometrical functions for $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, the table of derivatives, the table of antiderivatives. Local functional tables should not be given to the students in the finished form. High school students must be able to substantiate every meaning of the elements of such a table. Global tables include the information about the meaning of function for rather a great quantity of meanings of the argument. The main requirement in familiarization with the global tables is the possession of the potentially possible technologies of constructing them based on calculating skills, possessed by the students.

Graphic is a peculiar visual model of the function. N. A. Tarasenkova[5] (2002), analyzing psychological and semiotic characteristics, separates *contents and graphical interpretations of the function*, *abstract sketches of the graphics of the function* and *schematic graphics of the concrete functions*. Contents and graphical interpretations of the functions are constructed as graphics of concrete functions, carrying definite information about their properties. This information can be completed by the knowledge about the analytical setting of the function, the meaning of functions in the definite points (for example, in zeroes, points of extremum, overhang and others). Abstract sketches of the graphics of the functions include the information about the most important characteristics of the functions or classes of functions. And, finally, schematic graphics reflect the definite properties, necessary for solving a certain problem.

If $f: G \to R$, where $G \subset R$, that is f − the numerical function, then its graphic is defined as: $\Gamma_f = \{(x, y): x \in G, y \in R, y = f(x)\}$ That is, as a mathematical notion, from positions of the theoretical and set conception, the graphic of the function is equal with the function itself through binary relation. From the mentioned above the conclusion can be made that one can speak about the graphic as a purely mathematical notion and as the visual model, reflecting definite properties of functions. That's why it is rational to introduce a separate notion − *visual model of the function* (VMF). As any VMF does not include complete information, so to get data from VMF one should have some *subtext*. It can be, for example, the information about the belonging of the function to the definite class, additional information about definite properties of the function. Without this, it is impossible "to

[5] Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics: monograph*. Cherkasy, Ukraine: Vidlunnya-Plus (p. 219) (in Ukr.).

read" the simplest graphics. So, if $\hat{\Gamma}_f$ is VMF for the function f, $J_0(f)$ is some known information about f, then $\overline{J}\left(\hat{\Gamma}_f, J_0(f)\right)$ is the information obtained from the visual model of the function and subtext.

It is clear that one cannot build a static VMF (graphic-picture) for any function. There are examples like Dirichlet or Van der Waden functions. But there are also possibilities to construct VMF for such functions. First of all, these are dynamic models that are most effective in using information and communication technologies. Thus, one can visualize an absolutely continuous, but undifferentiated function through constructing a sequence of approximations, which will give a definite notion of the Van der Waden function's property. There is a possibility to observe the function's graphic with the infinite number of extremums around some point with the help of computer dynamic visual model (with the successive increase of the scale). Another possibility of the VMF actualization is the application of *plastics*. N. A. Tarasenkova[6] (2002) introduces plastics as a separate nonverbal sign and symbolic means in teaching mathematics. Thus, it is very hard to imagine a function's graphic of Dirichlet, but the leaps of the end of the pointer from line $y=1$ to line $y=0$ can give a notion of the function. It is proved by experiment, that the efficiency of the use of the visual models depends on the sex of the students[7] (Forster, & Mueller, 2002) as visual, intuitive explanations are more available for boys.

One of the most important means of mathematics is problems. As N. A. Tarasenkova[8] (2002) underlines, "they (problems) serve both the object of investigation and the means of teaching, they perform instructional, developing, educative and controlling functions". The problem as an object and means of teaching mathematics has been

[6] Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics: monograph.* Cherkasy, Ukraine: Vidlunnya-Plus (p. 236) (in Ukr.).

[7] Forster, P. A., & Mueller, U. (2002). What effect does the introduction of graphics calculators have on the performance of boys and girls in assessment in tertiary entrance calculus? *International Journal of Mathematical Education in Science and Technology, 33, 6,* 801-818

[8] Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics : monograph.* Cherkasy, Ukraine: Vidlunnya-Plus (p. 109) (in Ukr.).

pointed at by N. K. Rusin[9] (1980). Traditionally, all the problems are divided into four large groups: problems on calculation, problems on proof, problems on construction, problems on investigation[10] (Slepkan', 1978). Of recent an attempt to broaden the classes has been made. So, in the study of S. V. Muzychenko[11] (2004) the notion of the constructive algebraic problem is introduced: "By *constructive mathematical problem* we understand the requirement to construct a new mathematical object, which would satisfy the given conditions, by the means pointed visibly or invisibly by some given mathematical objects within limits of a certain theory"[12] (Muzychenko, 2004). S. V. Muzychenko understands the constructive problem as the generalization of traditional geometrical problems on construction and considers such types of constructive algebraic problems: problems on the construction of analytical objects, problems on the construction of graphic objects, problems on the construction of table and textual objects. By analogy with constructive problems, one can broaden the notion of the problem on calculation, and namely, *to understand as problems on calculation the problems on finding all the objects, satisfying certain correlations*. These objects can be numbers, sets (traditional understanding), functions, relations, operators, functionals, transformations of space, figures, and others. In such an understanding the problem P_1="Give an example of function $f: R \to R$, different from the equal constant, such, as $|f(x)|=1 \quad \forall x \in R$" is the constructive problem, because it requires constructing one concrete example, and the problem P_2="Find all the functions $f: R \to R$, different from the equal constant, such, as $|f(x)|=1 \quad \forall x \in R$" is the problem on calculation. Consider two more problems: P_3=

[9] Rusin, N. K. (1980). A problem as an object and a means of teaching mathematics. *Mathematics in school, 4*, 13–15 (in Rus.).

[10] Slepkan', Z. I. (1978). *Methodology of teaching algebra and accidence of analysis*. Kyiv, Ukraine: Radyanska shkola. (in Rus.).

[11] Muzychenko, S. V. (2004). *Constructive pronlems as the means of development of the creative thought of pupils in the process of teaching algebra* (Doctors' thesis). Chernihiv pedagogical university. Chernihiv. Ukraine. (In Ukr.).

[12] Muzychenko, S. V. (2004*). Constructive problems as the means of pupils' creative thought development in the process of teaching algebra* (Doctors' thesis). Chernihiv pedagogical university. Chernihiv. Ukraine. (p.9) (in Ukr.).

="Prove, that there exist functions $f: R \to R$, different from the equal constant, such as $|f(x)| = 1 \quad \forall x \in R$"; P_4 ="Do there exist functions $f: R \to R$, different from the equal constant, such, as $|f(x)| = 1 \quad \forall x \in R$".

Here, obviously, P_3 is the problem on proof, P_4 – that on investigation. It is easy to see, that the four problems described above, are interconnected by some relations. So, P_2 generalizes P_1, P_1 specifies P_3, which, in its turn, specifies P_4. By analyzing the systems of instructional problems, Y. M. Kolyagin, V. F. Kharkovska, & V. G. Hulchevska[13] (1979) consider the relations between problems as system-forming connections. Most often there occur the relations of the general idea, specialization, generalization, analogy, specification, modelling, limit case, and others.

By analyzing the system of problems of the functional contents line, we see the necessity of broadening four main groups of problems[14] (Slepkan', 2000). Traditionally, problems on constructing graphics are considered as elements of the class of problems on construction. But problems on construction are the problems of constructing ideal mathematical object, including geometrical ones. The main in the problems on construction is the sequence of the actions of constructing, a figure is only an illustration. As it has been settled above, by constructing a graphic we, as a rule, understand the constructing of some visual model of the function, thus the problem acquires a different content. It is not entirely clear which class of problems should those problems in which, according to the graphics, one should find a formula, defining the function, be referred to. Yet, these problems have one thing in common – the transformation from one way of defining the function to another. Analogous issues are characteristic of a series of problems of the contents

[13] Kolyagin, Y. M., Kharkovskaya, V. F., & Gulchevskaya V. G. (1979). *About the system of the teaching problems as the means of development of the mathematical thought of pupils. From the experience of teaching mathematics in the secondary school : manual for teachers*. Moscow, , Russia: Prosveshchenie, 114–118 (in Rus.).

[14] Slepkan', Z. I. (2000). *Methodology of teaching mathematics : manual for the students of the math. specialities of ped. teach. establishments*. Kyiv, Ukraine: Zodiac-EKO. (p.84) (in Ukr.).

line of the functions: transformation from the recurrent way of defining to the defining by a formula and vice versa, from the table way of defining the function to the analytical etc. To our mind, the problems of the mathematical modelling, in part – those connected with the functions, should make up a separate group. Traditionally, such problems refer to the problems on calculation. But the problems of modelling include important issues of formalization and interpretation of the results, it distinguishing them from the problems on calculation. Thus, we single out such groups of problems, connected with functions:

1) problems on calculation;
2) problems on transformations of the way of setting the function;
3) problems on the construction of functions;
4) problems on the proof of the functions' properties;
5) problems on the study of the functions' properties;
6) problems of the mathematical modelling by means of the theory of functions.

To the *problems on calculation* we refer:

- problems on finding images (the meaning of the function in the point, the image of the given set, the image of the figure in transformation);
- problems of finding pre-images of the functions;
- finding theoretical and set characteristics of the functions (the area of the definition, a set of meanings);
- finding "important" points of reflections (static points, critical points etc.);
- finding a set of the area of the definition with definite properties (blanks of monotonousness, convexity etc.);
- finding the meanings of the functionals (a rather wide circle of problems, for example, the meaning of the derivative in a given point, the maximum meaning of the uninterrupted function in a line, the least positive period of the uninterrupted periodic function);
- finding the meanings of the operators (the operator of differentiating, the definite integral, the primary (multi-semantic operator);
- finding the functions satisfying certain functional interrelations (functional equations, non-equations).

Transformation problems of the ways of function's setting. Unlike the work cited[15] (Muzychenko, 2004), where the problems of constructing graphics by the analytical record and others, refer to the constructive ones, we introduce a separate group of transformation problems. S. V. Muzychenko considers the transformation of the analytical record, tables, graphics, and the descriptions one function to be a generalized problem on construction, a constructive problem. We refer such tasks to a separate group, as *we consider, a formula, a graphic, a table are the sign and symbolic images of one object – the function.*

We will consider the issues of setting (defining) functions separately. If in the function the domain of the definition and the image are real numbers, we deal with the *real function*[16] (Kudryavtsev, 1985). These functions in general, are defined in the courses of algebra and the accidence of analysis. The simplest generalization of the real function is real function from several natural variables. It is clear, that the function is considered to be set if the domain of the definition, the image and the law of the correspondence are set, it correlates every element from the domain of the definition with one and only one element from the range of values. Traditionally in the methodological and instructional literature three ways of setting the function are distinguished – the analytical (formulae-backed), the graphical, and the table[17] (Slepkan', 2000). These are intuitively understood ways and they are adequately perceived even by the 7th graders. At the same time, the ways of setting functions assume a more careful analysis. In the article by L. D. Kudryavtsev[18] (1985) only the analytical method is highlighted as a precise one, which itself presupposes three ways. Thus, the setting of the function is separated directly with the help of formulae, it presupposing the separation of some fundamental class of

[15] Muzychenko, S. V. (2004*). Constructive problems as the means of pupils' creative thought development in the process of teaching algebra* (Doctors' thesis). Chernihiv pedagogical university. Chernihiv. Ukraine. (In Ukr.).

[16] Kudryavtsev, L. D. (1985). *Function. Mathematical encyclopedia* [in 5 v.]. V. 5. Moscow, Russia: Soviet encyclopedia, 712–720 (in Rus.).

[17] Slepkan', Z. I. (2000). *Methodology of teaching mathematics: manual for the students of the mathematics specialties of the pedagogical institutes.* Kyiv, Ukraine: Zodiak-EKO. (p. 235) (in Ukr.).

[18] Kudryavtsev, L. D. (1985) *Function. Mathematical encyclopaedia:* [in 5 v.]. V. 5. Moscow, Russia: Soviet encyclopedia, 712–720 (in Rus.).

functions, with which arithmetic operations, composition operations and the actions, connected with limit transformation are performed. Then the way of setting the function, being set by several formulae or a description is informed. For example:

$$f(x)=\begin{cases}2x,\ if\ x>0,\\ x^2,\ if\ x\le 0.\end{cases}$$

It is easy to see, that setting functions by description directly confines to setting by formulae. Thus, the function from the previous example can be written as: $f(x)=2x\chi_A(x)+x^2\chi_{\bar{A}}(x)$. Here χ_F is the characteristic function of the set *F*, *A* is the set of real positive numbers. To the analytical ways refer also the implicit setting of the function, that is by the equations as $F(x,f(x))=0$.

To our mind, the approach described above does not encompass all the strict methods of setting functions in school mathematics. The strict ones are such ways of setting functions, in which the function is determined unequivocally. Thus, by the table way one cannot unequivocally set the function with an infinite domain of definition. At the same time, for the function with finite area of the definition, the table can describe the function unequivocally. Further, we shall dwell only on the numerical functions with the infinite domain of definition.

The simplest attempt to describe the construction of functions by the "formulae" is by using the recursive procedure of the construction of the terms in the logical and mathematical languages (Kolmogorov, & Dragalin[19], 1982, Vereshchagin, & Shen'[20], 2002). Thus, one can introduce the functional signature $\Omega=\langle Srt,Cnst,Fn\rangle$. Here Srt is a non-empty set of *sorts of values* (sorts of individuals). $Cnst$ is the set of constants. Every constant can be of a certain sort. *Fn* is the set of fundamental functions (elements). With every

[19] Kolmogorov, A. N., & Dragalin, A. G. (1982). *Introduction to the mathematical logics*. Moscow, Russia: Publishing of the Moscow State University (in Rus.).

[20] Vereshchyagin, N. K., & Shen', A. (2002). *Languages and calculus*. Moscow: MCNMO (in Rus.).

fundamental function $f \in Fn$ one can connect the kind of a fundamental function, that is the expression $(\pi_1,...,\pi_k \to \pi)$, where π_k, π are the sorts of the functional signature, $k > 0$. We build functions (elements) of the signature Ω inductively. First of all, every function (element) of the signature will be put into some sort of correspondence (the sort of meanings of the corresponding function). We will separate a sort for the construction of numerical functions – real (complex) numbers (π_0). Other sorts are the types of numerical functions from several variables. The fundamental function, the sort of meanings of which defines some function, will be called *the fundamental operator with parameters*. Now we will inductively determine the elements of the functional signature Ω:

1) each function $f : R \to R$ of the kind $f(x) = c$, where $c \in Cnst$ is an element of the signature Ω;

2) the function $f_0(x) = x$, where $x \in R$ is an element of the signature Ω;

3) each fundamental function is an element of the signature Ω;

4) if f is the fundamental function as $(\pi_1,...,\pi_k \to \pi)$ of the functional signature Ω and t_1 is an element of the signature of the sort π_1, ..., t_k is an element of the signature of the sort π_k, then $f(t_1,...,t_k)$ is an element of the type π signature Ω.

It is the type π_0 elements that determine *numerical functions of the signature* (here even "fictional functions" are assumed, that is those with an empty area of the definition). It should be noted, that the definition of inverted functions (one can introduce the operator of the construction of the inverted function) corresponds to the given scheme.

In such an interpretation of the setting of the functions by the formulae, there arises a logical question about the set of fundamental functions *Fn*. One can ascribe it to the fact that they are also formed from some signature, but then the corresponding chain must end somewhere. Thus, naturally strict methods of setting the functions presuppose not only formulae. First of all, one can speak about the implicit setting of the functions in the wide understanding (called generalized implicit way – GIW). And namely: if in some formal logical and mathematical language the expression $T(x, y)$ with two free variables is built, then in the interpretation of Φ it sets the function, if, first of all, for every meaning x there exists not more than one y, such, as $T(x, y)$ is true, secondly, there exist x, y, for which $T(x, y)$ is true. Thus, generalized implicit way presupposes the proof of correctness that is the proof of the existence and the unity. A generalized implicit way of the introduction of the functions is natural for the high school mathematics. Thus, for example, the trigonometric functions are introduced. Also, the setting of the inverted functions can be described in the implicit way.

If there exists an algorithm, setting $y = f(x)$ in correspondence to every possible meaning of x, one can speak about the algorithmic setting of the function. In some studies[21] (Subbotin, & Bilotskii, 2008) an attempt of the classification of functions from the point of view of their algorithmic setting is made. The functions of the natural argument set by an algorithm and acquiring natural meanings, are called calculating. Each of such functions can be defined in one and the same signature, generating the so called partially-recursive functions. According to the generalized thesis of Cherch, the classes of the calculating and partially-recursive functions coincide[22] (Katlend, 1983).

Another way of defining functions is the axiomatic one. Here, two possibilities are meant. First of all, some functions can be introduced in the axiomatic theory itself, as, for example, the successor function in the formal arithmetic of Peano. The second option is as follows. Some *functional interrelations* are described, then the existence and the unity of functions satisfying these interrelations are proved. For example, the functions $\sin x, \cos x$ can be defined as the

[21] Subbotin I. Ya., & Bilotskii N. N. (2008). Algorithms and Fundamental Concepts of Calculus. *Journal of Research in Innovative Teaching Publication of National University, 1, 1*, 83-94

[22] Katlend, N. (1983). *Computability. Introduction to the theory of recursive functions*. Moscow, Russia: Mir (in Rus.).

functions, for which the correlations: $0 < \sin x < x < \frac{\sin x}{\cos x}$ with a rather little positive x, $\sin^2 x + \cos^2 x = 1$;

$\sin(x+y) = \sin x \cos y + \cos x \sin y$; $\cos(x+y) = \cos x \cos y - \sin x \sin y$

are performed. The research[23] **(Shilov, 1969)** proves the unity and the existence of such functions. Power, logarithmic, and exponential functions can be axiomatically described in a rather plain way.

One can also distinguish a whole group of ways of defining functions, connected with the transformation to a wider area of the definition. That is, to determine the function f on the set S, we use the defined function f_0 on the set $S_0 \subset S$ so, that $f(x) = f_0(x)$ $\forall x \in S_0$. Thus, for example, one can gradually define an exponential function on the chain of the sets: $N \subset Z \subset Q \subset R$. The last enclosure presupposes the operation of the limit transformation. We will call such a group of ways the ways of broadening the area of the definition (WBAD).

Among the ways of broadening the area of the definition a specific place is taken by the cases of the *automatic* continuation of the function on the more wide area of the definition. Let the class of functions F_W with the area of the definition W and the class $\grave{\mathrm{E}}_{\mathrm{W}}$ of subsets of set W be given. We consider that the pair $\langle F_W, \grave{\mathrm{E}}_{\mathrm{W}} \rangle$ are governed by the theorems of continuation, if with any f and g with F_W from the equality f and g on some set $T \in \Theta_W$ ($T \subset W$), there follows the equality f and g on W. Thus, in such a case functions expand on a wider area. The idea of functions extension is actual in the modern analysis. In the high school mathematics one can give the following examples. The polinomial of power n, given on an finite set with point $n+1$ automatically unequivocally continues on all the numerical line, any continous function, given on the set of

[23] Shilov, G. E. (1969). *Mathematical analysis. Functions of one variable. Parts 1-2*. Moscow, Russia: Nauka (in Rus.).

rational numbers Q unequivocally continues on the set of real number R, etc.

Inside the class WBAD one can differentiate the inductive ways of setting functions (ISF). The simplest version of ISF can be briefly described in the following way. Let there exist a chain of sets $G_0 \subset G_1 \subset ... \subset G_k \subset ...$ put one into another and the operator Γ, putting into a correspondence to the function f_k, defined on the set G_k, its continuation f_{k+1} on the set G_{k+1}. Then the function f_0 unequivocally defines the continuation on $\bigcup_i G_i$, where $i \in N \cup \{0\}$. Evidently, the case of ISF is the recurrent way of setting sequences.

In Figure 2.4.3 a scheme of defining numerical functions is presented.

Setting the function through the table (with infinite area of the definition) does not belong to strict ways of setting the function. If it is known, that the function is defined on a certain numerical segment and the table of meanings of

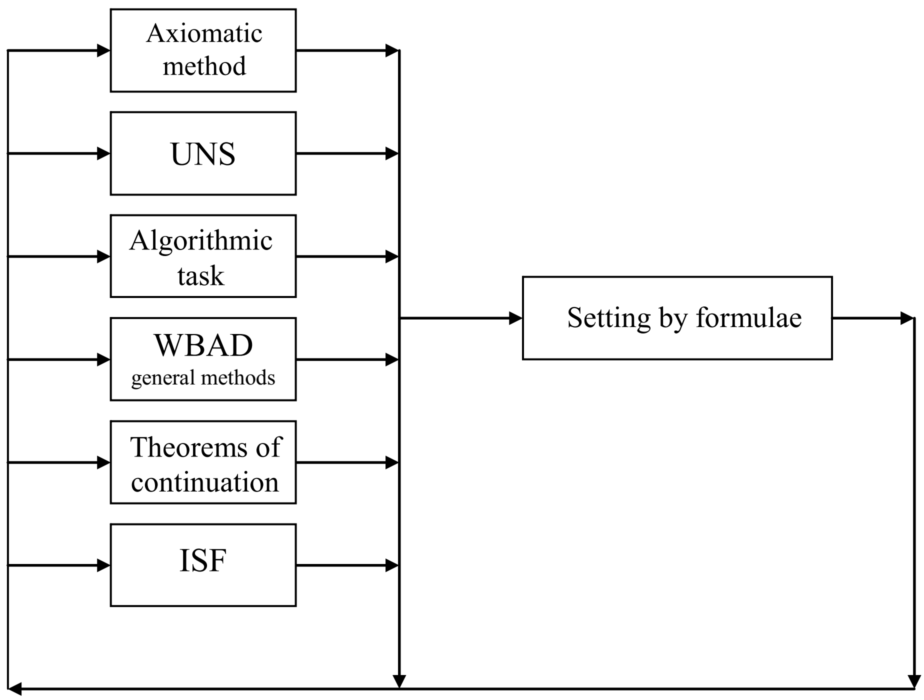


Figure 2.4.3 The scheme of the defining of numerical functions.

the function is given, then there arises the *interpolarization* problem[24] (Kudryavtsev, 1985). In school practice, especially in classes of physics, geography, and biology students often have to solve the easiest problems of interpolarization, but, as a rule, this process takes place without any substantiation that error will be rather moderate, non-critical.

Now we can give the corresponding definitions. Thus, to the problems of transformation of the way of setting the function we will refer the problems of transforming the strict ways of setting the function into any other way, non-strict ones included. The transformation from non-strict methods to strict ones in some cases refers to the *problems of constructing functions*. In table 1 problems of transformation of ways of setting functions are shown, here the way X (the first column) transforms to the way Y (the first line), that is $X \to Y$.

Problems on constructing a function. The definition of the constructive problem was given above. We do mean that the problems on the construction of a function should be constructive. In the contents line of functions such problems are the problems on constructing functions or function-related sets with certain properties. To construct a function means to set it in a strict way. The graphic here can only explain and illustrate ideas. Commonly accepted is following scheme of solving problems on construction: *analysis* $\to$ *constructive (algorithmic) description of constructing (which can be accompanied with an illustration)* $\to$ *proof* $\to$ *study*, as a rule, conditions, under which the construction is possible as well as issues of uniqueness are studied. If a problem offers to construct all the possible objects, satisfying these requirements, one can suppose that the given problem is a problem on calculation. The constructive problem presupposes the description of the setting of every way of solving. To the problems on constructing, the problems of interpolarization and approximation are referred.

The graphic of the function (the visual model of the graphic) can also be considered as the entity of some requirements to the function. To construct the corresponding function, first of all, one should "read the graphic", that is to move from the illustration to verbal and symbolic description, and then to construct the function.

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[24] Kudryavtsev, L. D. (1985) *Function. Mathematical encyclopedia* : [in 5 v.]. V. 5. Moscow, Russia: Soviet encyclopedia (in Rus.).

Problems on the proof of properties of the functions. Let us analyze possible types of problems on the proof concerning functions and their properties. In the problem on the proof one should prove the implication $X \Rightarrow Y$ (or two implications $A \Leftrightarrow B$ the problem then is split into two). Let us assume that the statement to be proved is as follows: $Y = A \wedge B$. Then, to prove this statement one should prove A, as well as B. That is in this case, the decomposition of the problem into two is made. The proof of the statement $A \vee B$ confines either to the reduction of the alternatives (to prove A or to prove B), or to the proof of the implication $\neg A \Rightarrow B$. This results from the evident tautology $(A \vee B) \Leftrightarrow (\neg A \Rightarrow B)$. Besides, it is known from the mathematical logics that any closed formula can be written in the anticipatory form: $Q_1 x_1 Q_2 x_2 ... Q_n x_n \varphi(x_1,...,x_n)$, where $Q_1,...,Q_n$ are quantors of universality or existence, $\varphi(x_1,...,x_n)$ is a non-quantor formula. If all the quantors are the quantors of universality, then the problem confines to the proof of several (maybe one) implications. Non-quantor implication $\varphi_1(x) \Rightarrow \varphi_2(x)$ in fact means enclose of one set to another. If $\varphi(x_1,...,x_n)$ is the atomary predicate, one can suppose, the problem is confined to the proof of some interrelation. If among the quantors $Q_1,...,Q_n$ turns out a quantor of existence, then the problem of existence will solve. Analysis of problems on the proof shows, among the implications very often implication $(T(x) \wedge T(y)) \Rightarrow (x = y)$ occurs. It is clear, it handles here about the unity. Thus, one can separate fundamental problems on the proof (in part – in studying functions): the proof of interrelations (for numerical functions – equations, non-equations, conditional equations and non-equations); the proof of enclosures; the proof of existence; the proof of uniqueness, combined problems on the proof. We will attract the attention to the fact that the proof of the existence can be constructive (the object is being constructed), reductive (the problem is confined to another problem of existence), significantly non-constructive.

Problems on the study of the properties of functions. Objects of the study can be numerical functions, transformation of the area, operators etc. We will consider *elementary problems on study,* problems on the study of the *first type*

Way Y / Way X	AM – axiomatic method	UNS –generalized implicit	AL – algorithmic	WBAD – the way of non-automatic broadening of the definition area	TC the use of theorems of continuation	ID – inductive definition	F – setting by a formula	G – the graphic way of setting	T – table
AM – axiomatic method	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
UNS – generalized implicit	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
AL – algorithmic	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
WBAD – the way of non-automatic broadening of the definition area	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
TC – the use of theorems of continuation	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
ID – inductive definition	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
F – setting by a formula	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*	Tp*
G – the graphic way of setting	K**	K**	K**	K**	K**	K**	K**	–	Tp*
T – table	K**	K**	K**	K**	K**	K**	K**	K**	–

* – problem on transformation of the way of setting a function;

** – problem on constructing a function.

Table 1. Problems on transformation of the way of setting and constructing functions.

and problems on the study of the *second type*. Elementary problem on the study we will call the problem, in which it is required to prove or disprove some hypothesis H. The example of formulation of such a problem: "Is the given function even?" The problem on the study of the first type is the problems with the suggested plan of the study, being the unity of calculated and elementary studying problems. An example of such a problem can be the problem formulated for the uninterrupted function: "To study the set function on its periodicity". Solving such a problem presupposes the proof or disproof of the hypothesis about the periodicity, and in the case if the function is periodic, you find its least positive period (the problem on calculation). In the problems on the study of the second type one should set a series of hypotheses oneself and prove or disprove them. Besides, new hypotheses can occur after the proof or disproof of some premature hypotheses and solving auxiliary problems on calculation.

Problems on mathematical modelling by means of the theory of functions. Usual textual problems, which, without doubt, are examples of problems on the mathematical modelling, traditionally refer to the problems on calculation[25] (Slepkan', 2000). Besides, problems on search and analysis of the functional dependences, occurring in the real situations, in the study of the natural, social, economic phenomena, can hardly be characterized only as calculating problems. Here, one can differentiate such groups of problems: those on the modelling of the functional dependencies, proceeding from some assumptions (the theoretical method); on the modelling of dependencies by the experimental data; combined experimentally-theoretical problems; interpretation and analysis of the models.

The separation of the main types of problems of the functional contents line enables us to choose fundamental (key) problems of the corresponding teaching topics. Fundamental problems must define all the system of problems through the entity of the main relations between problems. That's why *the necessary requirement in composing fundamental problems is the presence of all main six types of problems described above and their subtypes among fundamental problems*. Besides, the criterion mentioned does not determine on what contents

[25] Slepkan', Z. I. (2000). *Methodology of teaching mathematics : manual for the students of math. specialties of the ped. teach. establishments*. Kyiv, Ukraine: Zodiac-EKO. (p. 84) (in Ukr.).

the choice of the problems should be oriented. Very often the criterion in the choice of problems is their similarity with the problems of the state resultative attestation, entrance exams, tests, olympiads etc. That is, while choosing problems they orient only on the *conjunctural criterion*. This criterion should be taken into account while choosing problems, but it should not be determinative. From the positions of the competence approach, there must be problems in the system of fundamental problems, also selected by the criteria:

1) the application of the result of the problem, possible methods of solving in studying other subjects;
2) the practical application of the results of the problem;
4) the influence of the problem on the development of the pupil's outlook, his general culture;
5) propedeutics through the results and methods problems that can be presented in theorems and problems in studying the next themes;
6) training through the problems for the further introduction of complex notions;
7) the development of the technological component of the mathematical competency;
8) popularization of achievements of the mathematical science and its applications. As we see, the choice of the fundamental problems is a complex and multi-criterial issue.

Attention should be paid to the fact that, at first glance, a series of problems do not have any relation to the functional contents line, at a closer look, they are merely functional. Let's illustrate it by the example of combinatory problems.

Consider such a problem: "The number of a bus ticket consists of 6 digits. The ticket will be called lucky, if the sum of its first three digits equals the sum of the three last digits. Which bus tickets are more numerous: lucky or those, whose numbers can be divided by 11?" The formal solving is as follows: "Let's assume that H is the set of numbers of the lucky tickets, T is the set of numbers of tickets divided by 11, U is the set of all the numbers of tickets. Attention should be paid to the fact that $x=\overline{abcdef}\in H$, then $y=\overline{adbecf}\in T$.

Truly,

$$y=\overline{adbecf}=\ 100001a+1001b+1\ \ c+9999d+9\ \ e+(d+e+f)-(a+b+c).$$

Thus, $y=\overline{adbecf}=\ 100001a+1001b+1\ \ c+9999d+9\ \ e\vdots 1$. Next, one can introduce the mapping $p:H\to T$, such, as $p\left(\overline{abcdef}\right)=\overline{adbecf}$. It is easy to see that p is a one-to-one mapping, because if $p(x_1)=p(x_2)$, then $x_1=x_2$. At the same time, the number $291918\in T$, because it can be divided by 11, but there does not exist x, such, as $p(x)=291918$, that's why the number of elements H is less than the quantity of elements T. Thus, lucky numbers are less numerous, than those divided by 11". The idea of the reciprocally unambiguous correspondence is connected with the notion of equivalence of the infinite sets being considered.

It is useful to describe by the language of reflections the intuitively clear, but slightly formalized statements, which are often used in the extra-curricular work in mathematics. Thus, the discrete principle of Dirihlet (being considered in optional classes and partially in studying the topic "Fundamentals of the theory of divisibility")) can have the following formulation: "If $f:E\to W$ is the mapping of the finite set E with n elements to the set W with k elements, then there will be found the element $a\in W$, having not less, than $\frac{n}{k}$ pre-image and the element $b\in W$, having not more, than $\frac{n}{k}$ pre-image".

The constructions, necessarily considered in the optional classes are very important and, if possible, in studying the topic "Fundamentals of the theory of divisibility". Let S is the *finite* set. Consider the mapping $f:S\to S$. Let some sequence be set by the recurrent interrelation:

$$y_{n+1}=f(y_n).$$

Then it is easy to see, that (y_n) is the sequence, periodic, beginning with some number. Truly, as the number of the possible meanings of this sequence is finite, then two similar members will be found in it (the principle of Dirihle): $y_k=y_l, k<l$. Further, it is easy enough to prove the

periodicity of our sequence, beginning with number k. For example, suppose that one should prove that the last digits of the members of the sequence by Fibonacci are periodically repeated starting from some moment. Let φ_n be the last digit of the n-th member of the Fibonacci Sequence. Then, evidently, the recurrent interrelation is performed:

$$\varphi_{n+2} \equiv \varphi_{n+1} + \varphi_n (\mathrm{mod}\, 10).$$

Consider the sequence of pairs $\omega_i = (\varphi_i;\varphi_{i+1})$. We will introduce the set $G = \{(\alpha;\beta): \alpha \in \{0,1,2,...,9\}, \beta \in \{0,1,2,...,9\}\}$. It is obvious, that G is the finite set. Then we introduce the mapping $f: G \to G$ such as:

$$f((\alpha;\beta)) = (res_{10}(\alpha+\beta); res_{10}(\beta + res_{10}(\alpha+\beta))),$$

Here $res_{10}(\lambda)$ is the remainder in dividing λ by 10. Then we have: $\omega_{i+1} = f(\omega_i)$. As the number of pairs $\omega_i = (\varphi_i;\varphi_{i+1})$ is finite, there will be found k and p $(k < p)$, such, as $\omega_k = \omega_p$. It is from here the periodicity (ω_n) concludes, and, this means, our sequence (φ_n) too.

By analogy one can prove the above mentioned fact about presenting any rational number in the form of non-infinite periodic decimal fraction. Here the idea lies in the fact that the members of the sequence of the digits of the quotient depend only on the formed remainders, which can also be only non-infinite quantity. In its turn, every following remainder depends only on the previous.

It is very easy to show, whether the sequence (y_n) set by the recurrent interrelation $y_{n+1} = f(y_n)$, where for the finite set S the mapping $f: S \to S$ is *one-to-one mapping* (bijection), then the sequence (y_n) will be periodic from the beginning. It can be seen from the fact that if such sequence is periodic with the period t, beginning with the number m, then it is clear from the equality $y_m = y_{m+t}$, that $y_{m-1} = y_{m+t-1}$, $y_{m-2} = y_{m+t-2}$, $y_{m-3} = y_{m+t-3}$. Obviously, the inverted statement will not be correct. We have considered the corresponding issues in detail in the research paper[26] (Kirman, 2005).

At the beginning stages of studying the combinatorics inside the problems, one can introduce the notions of surjection, injection, and bijection for the second time. The simplest problems, where the terms figurate, can have, for example, such structure:

1. How many injections, mapping the set A with n elements into the set B with m elements do there exist? (there is a little trap in the problem – the case, when $m < n$).
2. How many surjections, mapping the set A with n elements into the set B with m elements do there exist?

In the theme "The elements of the combinatorics, the theory of probabilities and the mathematics statistics" we suggest systemizing the knowledge about the main combinatory compounds and characterizing them as mappings. Only then, the fundamentals of combinatorics will be put on the solid basis. It is useful to conduct the corresponding systematization in class in the form of "a definition game", where the students are offered to hold the explication of the known notions by the language of mappings. As a result, the following constructions will appear: let the basic set $S = \{a_1, ..., a_n\}$ is given. Then any mapping of the set $F_k = \{1; ...; k\}$ into the set S can be called the sequence. The r-permutation is the injections F_k into S. In case, when $n = k$, the permutations of settings from n elements are the bijections between F_k and S. The permutation of settings with repetitions is the surjection F_k *on* S. Here, the set of integer positive numbers is determined $(t_1, ..., t_n)$, the type of the change of settings where t_j is the quantity of prototypes of the element a_j, $j = 1, ..., n$. In the language of mappings, combinations are the mappings of the basic set S into the set of integer positive numbers. If we consider the mapping f where $f(S) = \{0; 1\}$, then we can speak about the combinations without repetitions. The combination with

[26] Kirman, V. K. (2005). The study of the periodical functions in advanced research of mathematics. *Didactics of mathematics: problems and investigations: intern. collection of research. papers, 24,* 281-287 (In Ukr.).

repetitions from the given n elements on m is the following mapping $f : S \to \{0; 1; 2;...\}$, that $f(a_1) + f(a_2) + ...f(a_n) = m$.

In general, one can point to an important role of the intellectual means of teaching in the competence-oriented studying of functions in physics and mathematics profile classes. It is manifested in the necessity to analyze the structure of the theoretical material, the arsenal of functions that the students are supposed to command, and in the specifics of the use of sign and symbolic means. Especially important is the realization of the role of problems as the management instruments in the background teaching mode.

CHAPTER 3. MATHEMATICAL TRAINING AT THE UNIVERSITY

3.1. Mathematical Schooling in the Ukrainian Higher Technical Education: Aiming at Fundamental Learning

Tetyana Krylova

Globally, the reform of the education system was mainly caused by the fact that the level of knowledge and skills acquired by graduates of higher educational institutions (HEIs) do not correspond to the society demand for specialists who would be competitive in the labor market[1] (Krylova et al., 2011).

Like any other country, Ukraine needs specialists who are not only capable of living and working successfully in today's environment, adapting to rapidly changing conditions, but who also are professional, competent specialists in their field, specialists who can think logically and abstractly, conduct research, and analyze the obtained results.

There is a growing need to enhance basic education, in particular mathematical, relying heavily on the unity of its natural sciences and the arts.

The terminological unit "enhancing foundation education" designates significant improvement in the quality of education and the educational level of people, in particular – shaping their scientific forms of systemic thinking by appropriately changing the content of educational disciplines and the methodology of educational process.

Improving mathematics education is facilitated by the activation and intensification of training, professional orientation of teaching mathematics, the use of interdisciplinary links, the introduction of active teaching methods, planning, proper organization, systematic management of students' unassisted learning and monitoring its outcomes[2] (Krylova et al., 2011).

[1] Krylova, T. V., Gul'esha, O. M., & Orlova, O. Yu. (2011, September). The conception of enhancing learning process of mathematics for students of higher technical school. *Methods of improving fundamental education in schools and universities. Proceedings of XVI Intern. scient.-method. conf.* (pp. 80-83). Sevastopol, Ukraine: SevNTU. (in Ukr.).

[2] Krylova, T. V., Gulesha, O. M., & Orlova O. Yu. (2011). Didactic principles of fundamentalization of mathematical education of students from non-mathematical specialties in universities. *Didactics of mathematics: problems and research: Internatoinal collection of scientific papers, 35,* 27-35. (in Ukr.).

The training of a highly skilled, competent specialist, ready to compete on the labor market has always relied and is increasingly relying on the comprehensive use of mathematics, given the rapid progress in science and technology. Currently, in the computerized and informatized environment, proper level of fundamental mathematical training for students of higher technical schools gains special value, because mathematics plays an important role in shaping such qualities of modern specialists as professional competence, creative thinking, and capacity for independent scientific research. Mathematics is the bedrock on which the learning of physics, astronomy, chemistry, general engineering and special sciences is grounded. Mathematical methods and mathematical modeling are widely used to solve practical problems in various branches of science, technology, economics, and manufacturing.

Taking into account a rather low level of mathematical competence of the new students of technical universities, it is vital to develop a complex of actions that will enable the teacher to manage the educational process, stimulate students' learning and cognitive development, demonstrate the importance and necessity of the responsible attitude to studying mathematics.

Encouraging learning and cognitive development of students

To make further social adaptation and unaided acquisition of specific practical knowledge and skills easier for a University graduate, there must be created proper conditions during the training.

Such skills can be developed by the student only on condition of dynamic cognitive and social involvement. This is facilitated by intensive, enhanced, and individualized learning.

Consequently, the encouragement of students' learning and cognitive development is an important element of teaching, in particular – in mathematics, oriented towards the acquisition of knowledge with the teacher's help, as well as via the independent search and acquisition of knowledge. «Enhancing the learning process is the improvement of methods and organizational forms of educational and cognitive work of students, which promotes active and independent theoretical and practical activities of schoolchildren in all parts of the educational process»[3] (Goncharenko, 1997). «Enhancing educational and cognitive activity of students refers to teacher-initiated mobilization of students' intellectual, moral and volitional, but also physical efforts to achieve specific goals of education, development and upbringing»[4] (Sliepkan', 2005).

Enhancing educational process in higher school has two components:

- enhancing teacher's activities (enrichment of scientific knowledge, pedagogical skills, content, forms and methods of teaching);
- stimulating students' involvement.

Enhancing teacher's activities aims at encouraging students' involvement, creativity, independence in the assimilation of knowledge, its application in educational activities.

Simulating students' involvement is aimed at improving the acquired knowledge, skills and abilities and at absorbing new knowledge.

The problem of learning enhancing is a long standing one. It has been discussed since Socrates' times. The stimulation of students' educational and cognitive activity, in higher mathematics in particular, is an important component of the University educational process.

The criteria for activating the educational and cognitive activity of students in mathematical disciplines is the formation of cognitive interest in mathematics, increasing involvement in the learning process, evidence of cognitive effort, the signs of autonomy in absorbing mathematics, the manifestation of cognitive independence, participation in student olympiads and conferences, independent search and use of mathematical methods to solve problems of interdisciplinary kind, professionally oriented tasks, as well as research tasks.

Students' educational and cognitive activity can be enhanced by sparking interest to the subject, through the professional orientation of learning, the visibility of learning results, the use of interdisciplinary connections, active learning methods, etc.

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[3] Goncharenko, S. U. (1997). *Ukrainian Pedagogical Dictionary*. Kyiv, Ukraine: Lybid. (in Ukr.).

[4] Sliepkan', Z. I. (2005). *Scientific foundations of teaching process in higher school*. Kyiv, Ukraine: Vyshcha shkola. (in Ukr.).

Diesterweg, F. A. wrote that «development and education can not be just given or transmitted to a person ... It is only the excitement that can be transmitted ... All the art of upbringing and education is no more and no less than the art of excitement»[5] (Sliepkan', 2005).

It is necessary to constantly stimulate students' interest in mathematics through keeping the involvement from lesson to lesson starting from the very first class. The professors should also emphasize that the saying «mathematics is the queen of sciences» is not just a figure of speech. It should be ppointed out that a vast collection of mathematical concepts and statements that reflect the properties of objects and phenomena of real world makes it possible to successfully solve various problems of science and technology relying on mathematical methods.

In studying differential equations it is worth noting that different physical phenomena by nature are described by the same differential equations. A number of tasks in physics, engineering and natural science are connected with motion (the path of the planet, the trajectory of an electron in an electron microscope and others), with the study of phenomena in a continuous medium (problems of the theory of elasticity, hydromechanics, aeromechanics), with the propagation of heat, diffusion, with aspects of electrostatics and the like.

It is wise to support classes with examples where life situations lead to problems solved by mathematical methods, which then helped to set a more general task and allowed to receive new mathematical notions (for example, the task to calculate the area of the curvilinear trapezoid gave birth to the notion of a definite integral).

The main universal directions of enhancing most traditional forms of mathematics education are its individualization and differentiation. Hence, enhancing students' learning and cognition in mathematics is facilitated by such progressive forms, methods and educational events as:

- professional orientation in teaching mathematics,
- mathematical modeling,
- independent work of students,

[5] Sliepkan', Z. I. (2005). *Scientific foundations of teaching process in higher school*. Kyiv, Ukraine: Vyshcha shkola. (in Ukr.).

- applying information and computer technologies in practical and laboratory classes,
- individualization and differentiation of training,
- simulation business games;
- simulation exercises;
- credit-module assessment of the acquired knowledge and acquired skills,
- student olympiads, competitions, scientific and practical conferences,
- technical means of training (educational films, slides, graphics slides, educational television, personal computers, tables, packages, models, etc.).

To boost the cognition of analytic geometry, we use the type of lecture where students independently prove some formulas on the blackboard, such as the equation of circle, sphere, hyperbola by their definition.

To activate the cognition of mathematical analysis we use the type of lecture where students prove some theorems on the blackboard, such as the theorem on continuous decreasing function after the teacher proves the theorem of continuous increasing function, and so on.

Learning and cognition can also be encouraged by challenging questions. See an example. The question could be: «Is it possible to prove the Cauchy theorem about the ratio between the gains of two functions, using the Lagrangian theorem on finite gains?»

Thus, it can be argued that the main strategic objective training enhancement is not an increase in the amount of knowledge offered to students, not compression of the knowledge provided or speeding up reading processes, but creating didactic and psychological conditions for conscious learning, getting students involved at the intellectual level as well as at the level of personal social activity.

The professional orientation of teaching mathematics in university education

Mathematical disciplines are the basis of mathematical training of prospective specialists. It is generally acknowledged that mathematics teaching and its study in a technical university, alongside with the general tasks of fundamental education, must be designed to enrich the course selected by the students. Consequently, the teaching of mathematical disciplines in a technical university should have professional orientation, be more practical, and closely connected with industrial production.

Prominent scholars, outstanding educators of the past, A. N. Krylov, M. V. Ostrogradsky, P. L. Chebyshov would focus on it that successful assimilation of mathematics is impossible without purely abstract teachings and argued the importance of mathematics for the understanding of surrounding world.

Soviet mathematicians and course developers A. D. Myshkis and B. O. Solonouts, called on creating production-oriented versions of the course in higher mathematics in technical universities.

Famous mathematician and educator L. D. Kudryavtsev,emphasized the main methodological principle of teaching mathematics – the idea of increasing the level of practical orientation of the mathematical disciplines taught.

In 1996 there were identified the principles of professional orientation of teaching mathematics in a technical university, which included components of the mathematical training for students of non-mathematical courses in technical universities[6] (Krylova, 1998). In the study of mathematics the following equitable tasks should be solved:

- the introduction of professional orientation,
- developing in students a rational system of mathematical thinking, instilling mathematical culture;
- developing students' skills and abilities to model applied problems and solve them rationally;
- application of information and communication technologies.

We can often hear from students: «Why do we ever need mathematics?»

The complexity of mathematics as a subject is in its abstract nature. Blaise Pascal wrote that the subject of mathematics is so comprehensive that it is not only possible to make it engaging, but it is also necessary. In order to get the students interested, educators introduce methods of active learning, in particular, didactic games with professional orientation, etc.

[6] Krylova, T. V. (1998). *Problems of teaching mathematics in technical universities*. Kyiv, Ukraine: Vyshcha shkola. (in Ukr.).

Mathematics has serious and simultaneously relevant applications. The task of mathematics teachers is to reveal profound connection between the problems of societal life and the developments in mathematical thought in order to demonstrate to students repeatedly, and with a variety of examples, why and in what way the use of mathematical methods allows to solve practical problems, as well as in what way practical problems inevitably result in the need of further development of mathematical science itself and its methods. It is advisable to point out possible links between mathematics and other branches of knowledge and activities.

To help students develop proper idea of the origin of theoretical knowledg it is necessary to show how scientific theories are born from pressing tasks, to analyze the way basic concepts and ideas come out of common images. One example is the problem of calculating the area of curvilinear figure, which leads to the concept of definite integral, the calculation of mass of an inhomogeneous body, which brings us to the concept of triple integral, and so on. These aforementioned instances can help us prove that the concepts, for example, of definite, multiple, curvilinear integrals arose from the actual needs of people. Assigning adequate role to mathematical theories when solving practical problems will allow students to correctly assess the role of mathematical disciplines in society and in scientific paradigm.

What is the proper way to implement job-oriented approach to teaching mathematics?

Here are a few examples. When studying the topic «Derivative» along with the geometric and mechanical interpretation of the derivative it is also necessary to teach students interpret it not only as the change rate of one variable from another, but also give an example of such an interpretation of the derivative, which would reflect a certain notion that students associate with their future job. Thus, for students of the energy engineering faculty (groups ED (Electric Drive), ET (electrical engineering), ES (Electronic Systems) it is advisable to interpret the derivative, for example, as amperage at a given time.

For students of chemical sciences the derivative can be interpreted as the linear density of the material non-uniform line, and so on.

For the students of the economic profile the introduction to topic «Extreme values of two variables» should start with the statement of the problem: «What size should there be of a can of cylindrical shape and given capacity V, so that the least amount of material was spent on its production?» There is immense potential of professionally-oriented mathematics training in the tasks from sections «Differential equations» (compilation and solving Cauchy tasks, boundary value problems), «Equations of mathematical physics» and «Operational calculations» (construction of mathematical model for the given applied task, drafting equations, initial or boundary conditions, solving Cauchy task or boundary problem) and others.

All said above doesa not only demonstrates the potential of mathematical apparatus, but also encourages students to study mathematics, expands their horizons, contributes to overall and mathematical development of youth.

The introduction of professional orientation in mathematics training is a way to eliminate the existing contradiction between the demand in society for skilled specialists and the current state of mathematical training of students in technical courses.

It should be admitted that mathematical education of students quite often begins and ends in the courses taught by the departments of mathematics. The application of mathematical methods in special courses, even if does take place, is almost unrelated to the general course of mathematics and doesn't contribute to the common goal to develop mathematical thinking and form mathematical skills in engineers.

To strengthen the applied orientation of mathematical courses, mathematics teachers should maintain constant scientific and methodological contacts with specialized departments, use mathematical tasks and terminology of those departments in teaching mathematics, take part in scientific, state-funded and economic contract work of special departments. In a technical university, mathematics should be oriented towards the course specifically chosen by the student, along with the general objectives of fundamental education, which it solves.

The task of preparing a specialist capable of using computer technology and mathematical methods systematically in their practical work can not be solved only by the efforts of mathematics departments. This is due to the limited time students can contact with such departments, as well as because of inevitable limitation in the scope of application for mathematical methods in the courses of these departments. Therefore, it is vital that all departments involved in the training of engineers consistently and specifically use modern computing methods to solve various tasks»[7] (Krylova, 1998).

In a competency-oriented educational paradigm, professionally directed teaching of mathematical disciplines to undergraduate students of technical universities is a prerequisite for the formation of their basic professional competences.

Visuality in teaching mathematics

Proper use of visuality for students as a source of new information, illustration of information, visual support to introduce new concepts is a one of the means to enhance training.

A renowned Swiss educator Y. G. Pestalozzi was attaching great value to visuality in teaching.

A well-known Czech educator and humanist Ya. A. Komensky considered visuality the golden rule of study.

Visual support of learning promotes mastering of knowledge by the students through direct contemplation, observing objects, phenomena, processes, through embodied cognition.

According to Komensky, visuality presupposes the following:

- real objects and their direct contemplation,
- models or copies of objects;
- image of an object or phenomenon.

To enrich mathematics teaching with visuality it is important to use graphs, tables, drawings, models, etc. It is advisable to locate the means of visuality in mathematics classrooms, subject classrooms, in rooms where mathematics classes are held.

Interdisciplinary links of mathematics with other disciplines

One of the ways to enhance and stimulate the learning process, to boost students' learning and cognition is to rely on interdisciplinary links of mathematics with other academic disciplines.

[7] Krylova, T. V. (1998). *Problems of teaching mathematics in technical universities*. Kyiv, Ukraine: Vyshcha shkola. (in Ukr.).

Interdisciplinary links are the development of basic concepts from common scientific theories and laws that are studied in classes on related disciplines with the aim of mastering the theory comprehensively.

Methods of applying knowledge from other subjects can be determined via careful study of curricula, plans and course materials for related disciplines[8] (Krylova et al., 2011).

In teaching mathematics, interdisciplinary connections perform methodological, educational, developmental, and constructive functions.

The methodological function implies that it is only through interdisciplinary links that students can develop a holistic view of the world, which was stressed by academician V. I. Vernadsky in his day.

The educational function promotes the formation of such qualities of knowledge acquired in mathematics classes as systemicity, depth, awareness, flexibility, which facilitate assimilation of relationships between mathematical and general concepts.

The developmental function is determined by the role of interdisciplinary connections in shaping independent and creative thinking of students, in the formation of their cognition, autonomy and interest in learning mathematics.

The upbringing function implies that interdisciplinary links promote comprehensive approach to the education of young people while studying mathematics.

The constructive function lies in the fact that with the help of interdisciplinary connections the teacher improves the content of course materials, methods and forms of teaching.

The most transparent interdisciplinary connections of mathematics are with physics, theoretical mechanics, special disciplines, which rely on series, differential equations, etc.

Interdisciplinarity is a modern teaching principle that influences the selection and structure of course material for a number of disciplines, enhances the systemicity of the acquired knowledge, activates learning, focuses on the application of complex forms of learning management, ensures unity of the educational process.

[8] Krylova, T. V., Gulesha, O. M., & Orlova O. Yu. (2011, April). Interdisciplinary links of mathematics with other disciplines in teaching mathematics to students of technical universities. *Mathematical problems of technical mechanics – 2011. Proceedings of intern. scient. conf., 2*, (pp. 132-133). Dnipropetrovsk – Dneprodzerzhinsk, Ukraine: DSTU. (in Ukr.).

Work at problems of interdisciplinary content, applied, professionally oriented and research tasks boosts students' involvement and autonomy in the study of mathematics.

Let us formulate the requirements for the compilation of these tasks: the task should be correctly presented by the teacher, be clear to the students, be manageable, it should engage them due to an interesting formulation, unusual presentation of the question or the process of solving, the task should as well develop a student's life experience, show the opportunity to use acquired knowledge in some life situations.

The tasks of interdisciplinary content, professionally directed research tasks fulfill the following functions:

- educational (the use of these tasks is aimed at forming a system of knowledge, skills and abilities for students at different stages of mathematics training),
- developmental (solving such tasks develops the skill of comprehending the results obtained, making appropriate generalizations, comparisons, conclusions),
- upbringing (upbringing of future specialists can be carried out via specified tasks in mathematics classes),
- assessment (these tasks are educational).

Solving these interdisciplinary tasks helps shape and improve basic, subject, particularly, mathematical, professional and practical competences in students of technical universities.

Traditional and non-traditional teaching methods

One of the psychological and pedagogical conditions for boosting students' learning and cognition is «the dynamism, variety of methods, means, forms and tools of teaching and instructing, focusing on the development of students' active research activities, the priority of methods and forms of active learning»[9] (Kremin', 2008). «… the term teaching method (in practical aspect) signifies the modes of teachers' and students' work which help master the knowledge, skills and abilities, shape students' outlook, develop their capabilities.

Outstanding Russian mathematician N. I. Lobachevsky noted: «In mathematics it is teaching that matters the most».

[9] Kremin', V. H. (Ed.). (2008). *Encyclopedia of Education*. Kyiv, Ukraine: Yurincom Inter. (in Ukr.).

Teaching methods can be classified based on different grounds: the source of knowledge, kinds of activity, the logic of the educational process, etc.»[10] (Sliepkan', 2005).

Methods of training, according to one of the classifications, can be informational and informative, explanatory and illustrative, problem-based (problem, partly search, heuristic, research presentation of course material), logical, reproductive ones.

Methods of teaching can be divided into traditional and non-traditional.

Non-traditional teaching methods include methods of active learning (methods of developmental learning, «active» methods of teaching, etc.).

The idea of developmental education was put forward by outstanding educator Y. G. Pestalozzi.

An outstanding teacher K. D. Ushinsky offered the concept of developmental education in his book «Pedagogical Anthropology» (1837).

The problem of enhancing learning gave birth to the concept of «active learning». Active learning is a transition from regulatory, algorithmic, programmed forms and methods of organizing a learning process in higher education to developmental, problematic, research and search ones, which spark cognitive motives, interest in future professional activities in learning.

«Developmental training is focused specifically on the content, principles, organizational and methodological support of the educational process with the aim to achieve the highest efficiency in shaping schoolchildren's cognitive skills: perception, reflection, memory, imagination, creative abilities in various activities»[11] (Kremin', 2008).

The best-known systems of developmental education are general developmental didactic systems by D. Yel'konin, V. Davydov, V. Repkin, V. Sukhomlinsky.

The analysis of these systems permits to formulate general conclusion: developmental education is a learning process, where along with the study of course material students develop intellectually, their learning and cognition are aimed at the formation of a well-organized system of knowledge,

[10] Sliepkan', Z. I. (2005). *Scientific foundations of teaching process in higher school*. Kyiv, Ukraine: Vyshcha shkola. (in Ukr.).

[11] Kremin', V. H. (Ed.). (2008). *Encyclopedia of Education*. Kyiv, Ukraine: Yurincom Inter. (in Ukr.).

skills and abilities, at processing cognitive structures and operations within this system.

In late 1970s there were laid the foundations for active learning, relying on the research of psychologists and instructors of problem learning, mainly on the basis of school education.

«Problem learning is one of the types of developmental learning ... The essence of problem learning is in the students' search work...»[12] (Goncharenko, 1997).

The research[13] (Sliepkan', 2005) substantiates the need to include problem methods in all types of educational work of university students, it also argues that problem education most fully and adequately describes teacher-student collaboration.

The main representation of modular-developmental teaching technology in higher education is problem-modular lecture, which is an innovative form of training.

The main types of a problem-module lecture include scientific and informational, scientific and project, ideological-reflective lectures.

Heuristic and partially search modes are varieties of problem learning.

The main form of heuristic learning is a heuristic conversation. «Heuristic conversation is a question-answer form of instruction in which the teacher does not transmit formalized knowledge to the students, but making them come to understanding new concepts, conclusions and rules on the basis of their knowledge, representations, observations, life experience by skillfully asked questions, sometimes guided, which do not contain direct answer»[14] (Goncharenko, 1997).

With partial-search method of teaching the teacher formulates the problem, solves it almost fully and students are only partially involved in the search activity at its individual stages.

Quite autonomously from the concept of developmental education there were elaborated the so-called «active methods of learning», which encompass educational business games, case study, brainstorming, the method of immersion, the

[12] Goncharenko, S. U. (1997). *Ukrainian Pedagogical Dictionary*. Kyiv, Ukraine: Lybid. (in Ukr.).

[13] Sliepkan' Z. I. (2005). Scientific foundations of teaching process in higher school: Teaching aid. – Kyiv, Ukraine: Vyshcha shkola.

[14] Goncharenko, S. U. (1997). *Ukrainian Pedagogical Dictionary*. Kyiv, Ukraine: Lybid. (in Ukr.).

seminar-discussion, visits to manufacturing plants, field trips, mail sorting etc.

Business games belong to game-based learning, which boost the development of students' professional interest, sharpen the skills of independent work, the pace of their general and mental development intensifies as well.

Business game is «a sort of game where imagined situations shape the content of professional activity of future specialists»[15] (Kremin', 2008).

The purpose of educational games is to help students develop the skill of combining theoretical knowledge with practical activities, independently obtain the necessary information, acquire knowledge.

To use the method of case study a collection of detailed descriptions of situations is compiled. The student highlights features of a problem, important information, decides what needs to be further specified, finds out possible solutions to the problem, discusses various options with the teacher, independently chooses some solution to the problem. Of course, this is not equal to independent decision-making in real life, but it helps to acquire life experience.

This method can be recommended in a simplified form in mathematics teaching, for example, when solving indefinite integrals by the method of substitution.

The method of brainstorming is groupwork to solve a creative problem. Ukrainian universities have gained massive experience of its application. The method is used both as a way of solving the task and as a way of learning, since the knowledge and experience of the discussion participants become accessible for everyone and can be effectively absorbed when discussing the problem.

This method works well in solving complex problems of analytic geometry, tasks dealing with differential equations of initial or boundary conditions, and so on.

The method of immersion is an active teaching method with elements of relaxation, suggestion or game used in the teaching of humanities, in particular foreign languages.

Olympiads, competitions, scientific-practical conferences also relate to active training provided students prepare and participate autonomously.

[15] Kremin', V. H. (Ed.). (2008). *Encyclopedia of Education*. Kyiv, Ukraine: Yurincom Inter. (in Ukr.).

Both «developmental learning» and «active methods of learning» rely on the idea that the person's activity in education ensures the achievement of upbringing and learning, overall and professional development of the future expert's personality.

The use of information and communication technologies in studying mathematics

The use of computer technology is a powerful means of intensifying and boosting learning, mathematics in particular.

According to the information approach, learning is an individualized process of students' work with sign information on the screen. Recently, the contents of textbooks and tutorials are input into a computer. Although, if the course material was incomprehensible in the subject language, then it will remain unclear in the programming language.

In countries with extensive computerization experience scientists believe that real achievements in this field do not warrant that the use of computers will radically change the traditional system of training. Using a computer in a learning process is not a panacea.

There should be worked out fundamentally new learning tools, where the computer would fit naturally as a powerful learning tool.

The electronic environment is capable of shaping such qualities as the capability to experiment, flexibility, structuring etc., which facilitates further creative learning cognition, the establishment of links between new and old knowledge, and so on.

The use of computers should shape students' reflection, orienting it to the search for systemic relationships and regularities. Computerized training should rely only on the content, which cannot be absorbed without the computer. It should also be noted that improved students' involvement directs them to independent work, and systematic independent work on course material at classes and in extracurricular time boosts activity, that is, the activity and autonomy of the individual are closely interconnected and complement each other.

Management of independent work of students in studying mathematics and its assessment

The courses of mathematical disciplines are quite complex, and students are not able to master the course material without teacher's assistance.

The instructor must teach students to work with literature, to think logically. Pedagogically correctly organized independent work of students (IWS) gives the desired result with teacher's guidance. This kind of work with students requires a lot of intellectual, material and financial expenses.

Students' unassisted learning activities (preparation of small research papers, reports, individual hometasks, etc.) is the the main educational and cognitive work to promote active acquisition of theoretical knowledge and practical skills in mathematics.

The problem of managing students' unassisted work and monitoring its implementation is one of the most topical problems of higher education.

Organizing and managing students' unaided learning activities has always been one of the top objectives of university instructors, in particular teachers from mathematical departments. It is the study of mathematics that promotes not only mathematical but also overall development of a personality, the development of their logical and abstract thinking.

As stated in[16] (Kremin', 2008), «students' independent work is planned individual or collaborative work of students, performed according to the tasks and under methodological guidance of the teacher, but without teacher's direct participation».

Educators, methodologists, psychologists, teachers, professors in universities have been paying decent attention to solve the problem of schoolchildren's and students'independent work.

The results of scientific studies of psychologists and teachers show that it is possible to achieve high level of professional skills, competence and creativity only when the strife for self-development and self-improvement is realized. And this means that, students as future specialists should be given skills of independent work as a top priority.

[16] Kremin', V. H. (Ed.). (2008). *Encyclopedia of Education*. Kyiv, Ukraine: Yurincom Inter. (in Ukr.).

To achieve this, students' independent work must be thoroughly planned, organized and monitored.

Independent work is an obligatory part of students' learning and cognition, the foundation of higher education. Independent work is also a mode of learning under teacher's guidance and a way to encourage students to master methods of independent learning and cognition as well as to develop intellectual capabilities of the personality.

One of the tasks of higher education is to shape students' skills and abilities of independent work, the ability to independently manage the process of their learning. Hence, independent work of university students is an integral part of training a future specialist.

Oral questioning, various types of control papers, testing, modular tasks and the main types of assessment to check and evaluate the acquired knowledge, skills and abilities of students in mathematics.

To manage autonomous students' learning and cognition teachers must:

- define the goals precisely;
- facilitate the achievement of the goal through mathematics curriculum,
- ensure the achievement of certain goals of mathematics training in the process of studying and mastering each mathematical discipline by self-control, assessment and focused correction by the teacher.

According to the didactic goal, independent learning can be preparatory, training, summarizing and recapitulative, assessing or it can be aimed at obtaining new knowledge.

The most common types of independent work of students in mathematics are:

- work with textbooks, educational and teaching aids, didactic materials in order to understand and acquire new knowledge,
- studying on a personal computer,
- solving tasks, examples, in particular creative tasks,
- laboratory work,
- mathematical modeling,
- preparing short papers with elements of scientific

research for scientific, scientific-methodical and scientific-practical student conferences.

Independent fulfillment of tasks is the most reliable indicator of the quality of acquired knowledge, skills and abilities of the student.

It is very important to properly organize students' independent learning in mathematics, to monitor this work and control its implementation.

Among the requirements put forward for the organization of students' independent work, let us stress the following:

- correspondence with the purpose and tasks of studying and processing the material,
- thorough selection of content and amount of educational material by the teacher,
- the correspondence of types of independent work to real learning capacity of students,
- use of information-computer technologies and pedagogical tools,
- personalized system of independent tasks,
- adherence to the standards of volume of educational materials for students' independent study.

We suppose that before performing the first independent task students it is worth reminding and explaining to students one piece of wisdom: «You can not teach, you can help the person learn». It is necessary for students to realize the truth that independent work in its essence involves the utmost involvement of students, that they can learn only by working actively, by reaching certain conclusions independently in the process of studying the theoretical material and solving educational tasks. When organizing independent work it is necessary to convince students that they are capable, that they can manage, that is to inspire their confidence, it is important to encourage students to overcome the barrier of fear when they meet new theoretical material on mathematics, before solving formal, professionally directed and applied mathematical tasks.

Managing independent work is the most difficult stage of the class. In order to teach the student to work independently, a great deal of preparatory work is required, namely: the teacher must carefully select tasks and examples on this topic, prepare cards with differentiated tasks, consider the sequence of tasks, their variability, etc. The use of powerful computer learning tools and related software enhances the efficiency of independent student work. Students who systematically work independently with the computer learn to select, organize information, make conclusions, adapt more quickly in the new environment. The teacher must be aware of different level of students' educational and cognitive abilities. To come to this understanding, you need to know students' interests, to differentiate approaches to independent work with them, to respect each student's individual learning capacity in mathematics.

When managing independent work of students in mathematical disciplines, it is especially important to use methodical and didactic materials (textbooks, manuals, collections of tasks, methodical instructions).

In conducting independent work it is highly needed to have systematic feedback – this represents the teacher's control and self-control. Students must have a chance to improve their knowledge during the course, correct the mistakes made. The more accurate the information about the error or the wrong decision of the problem is, the more effective assistance should be given to students.

The degree of independence of students in individual tasks is an important indicator of their success in learning.

Carrying out independent work should not be of episodic, but of systematic nature, which helps to involve students in systematic work in the classroom and at home when solving individual tasks.

Independent work encourage students' active reflection, contributes to the development of their conscious attitude to systematic educational work.

Independent educational and cognitive activity develops in students the following personal qualities: autonomy, productivity, flexibility, initiative, attention, perseverance, endurance, critical thinking and other positive qualities. Thus, when carrying out independent work there should be unity of processes of «knowledge acquisition» and the development of «reflection skills».

The problem of managing students' independent work is complex and multifaceted and therefore needs due attention, new ways to solve it.

Unfortunately, first-year students can not work with educational literature on their own, they can not use the library catalog, they were not taught to recapitulate theoretical material on a repeated basis, nor systematically do homework, without any postponing for "later", etc.

Our research of this problem and many years of university experience make it possible to suggest the following steps for the organization of independent work of students in higher mathematics:

- to teach to work with textbooks and tutorials autonomously,
- to instruct to use the library catalog,
- systematically persuade that only self-acquired knowledge is strong, that it is necessary to systematically do homework,
- to involve students in independent work, step-by-step and continuously,
- at the very start of the course to offer students a detailed curriculum of the discipline for this semester, the timetable of assessment, indicating relevant topics, sections,
- to provide each student with packages of individual home and modular tasks, as well as methodical instructions, demonstrating typical solutions of mathematical tasks and tasks of varying complexity,
- to persuade students that they should systematically carry out self-control of their progress and results of their work, correct and improve their solutions,
- to diagnose the quality of acquired knowledge, developed skills and abilities of students in mathematical disciplines by means of questioning, testing, holding assessment, holding individual and modular tasks,
- to correct the educational process according to the results of diagnostics.

E-learning package (ELP) contains courses' curricula, student aids, test tasks of past years, homework assignments, lectures, instructions and comments to texts for students, etc. Through this system, you can also hold self-assessment.

The objectives of Ukrainian higher school to improve the quality of of professional training for students in technical sciences require to search for new forms and methods to organize overall educational process, including the management of mathematical training.

When organizing the educational process one must take into account that no two student groups are identical, just like there are no two identical personalities in their intellectual and overall development. Lectures and practical classes in higher mathematics, as well as in any other discipline this academic year can not be an exact copy of the lessons from the previous academic year. This means that the learning process should be adjusted. These adjustments depend on the scope of learning and students' learning capacity.

Identifying scope of learning and capacity to learn is an important component of the teaching process. It is necessary to diagnose students' scope of learning and learning capacity from the very first classes at universities, in particular from the very beginning of higher mathematics course. Diagnostics includes monitoring, verification, evaluation, accumulation of statistical data, their analysis, revealing dynamics, trends, forecasting further development of events.

To improve the learning process, diagnostics should be guided by the following objectives:

- preliminary identification of the level of students' knowledge, defining gaps in education, in particular in mathematics,
- confirmation of successful learning outcomes,
- planning the next learning stages,
- improving motivation by encouraging academic achievement, etc.

The most important principles to assess student learning level are objectivity, systematicity and visuality (publicity). It is advisable to control, check and evaluate students' knowledge and skills in mathematics in the same logical sequence in which the study is conducted. In order to diagnose the students' scope of knowledge in mathematics there can be carried out some control of the acquired knowledge, skills and abilities.

For the instructors to have an idea the kind of students in the group and to be able to plan curriculum according to their level, it is necessary to reveal their scope of knowledge in elementary mathematics first. In order to do this, they carry out "zero" test paper at the very first practical lesson on higher mathematics, namely a test paper to determine the residual knowledge of the freshmen.

Test papers to determine the residual knowledge of students are carried out at the beginning of each semester to identify gaps in their knowledge, abilities and skills in mathematics acquired during the previous semester, so that students could responsibly overcome these shortcomings.

Students' independent work is monitored systematically at each class in higher mathematics. It can be carried out via questioning at lectures and practical classes, short (5–10 minutes), traditional (45–90 minutes), «rector's» (90 minutes) tests, «express assessment", testing and defending individual homework and small research papers etc. All these types of control are evaluated in points according to the complexity of the task, then an integrated mark is presented.

An important factor in learning are the types of advancement (learning acquisition). The pace of learning progress is influenced by the student's potential capabilities, the scope of effective knowledge and generalization of thought, which are components of the concept of learning capacity.

Identifying the level of learning capacity is an important and painstaking job that requires teachers' effort and time. It should be noted that determining the level of learning capacity is much more complicated than determining the scope of knowledge. These should be tests made by educators and psychologists.

All these measures of pedagogical diagnostics and information about the students' scope of knowledge and capacity to learn mathematics empower teachers to plan the learning process in a way to enrich classes on higher mathematics with personalized learning tools in order to develop students' mathematical and general abilities, that is, to give teachers an opportunity to adjust the teaching process.

One of the forms of enhancing educational process is students' research work, planning, organization, leadership, motivation and control being the main principles of its effective management.

Research work is one of the aspects of independent work of students.

It is also possible to suggest interdisciplinary assessment measures for engineering students (for example, test paper, which includes questions on theoretical mechanics, material resistance and higher mathematics).

Thus, optimal combination of predominantly active forms and methods of students' independent work contributes not only to the mastery of mathematical disciplines, but also to the formation of skills for independent work in general, in educational, scientific, professional activities, the ability to take responsibility and come to constructive solutions independently.

The analysis of methods, organizational forms, measures and means of training in higher technical school, current state of mathematics teaching to students of non-mathematical specialties, as well as practical work at mathematics make it possible to state that activation of educational and cognitive activity of students, enhancement of training, professional orientation, use of interdisciplinary links, the introduction of active learning methods and «active» methods in the teaching process, students' independent work etc., significantly improve the quality of mathematical education of future specialists in technical sciences.

3.2. Computer-based Methods for Teaching Differential Equations to the IT Bachelor Students

Kateryna Vlasenko, Iryna Sitak

Education development strategies in Ukraine emphasize the priority of computer-based teaching in higher educational establishments. The application of computer-based methods in fundamental and professional training of bachelors in information technologies (IT) is consistent with the basic requirements of the educational and qualification characteristics of specialists, whose future professional activity is directly related to the involvement of computers for the development, application, and research of mathematical models of various processes. Moreover, regular development of IT and the intensification of their application in teaching by such foreign researchers as O. Alabi, A. Magana & R. Garcнa[1] (2015), M. Caprile, R. Palmén, P. Sanz & G. Dente[2] (2015), M. Castro, M. Reboredo & M. Fanovich[3] (2014), M. Goos, I. Hathaway, J. Konings & M. Vandeweyer[4] (2013), P. Mell & T. Grance[5] (2011, made it possible to use these technologies to organize the educational and professional activity of information technologies students while mastering differential equations (DE).

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[1] Alabi, O., Magana, A., & Garcнa, R. (2015). Gibbs Computational Simulation as a Teaching Tool for Students' Understanding of Thermodynamics of Materials Concepts. *West Lafayette: Purdue University, 37 (5-6),* 239-254.

[2] Caprile, M., Palmén, R., Sanz, P., & Dente, G. (2015). Encouraging STEM studies. Labor Market Situation and Comparison of Practices Targeted at Young *People in Different Member States.* European Commission, Brussels.

[3] Castro, M., Reboredo, M., & Fanovich, M. (2014). Incorporation of Information and Communication Technologies (ICTS) in Materials Engineering. *Argentina: University of Mar del Plata, 36 (1-2),* 1-10.

[4] Goos, M., Hathaway, I., Konings, J., & Vandeweyer, M. (2013). High-Technology Employment in the European Union. *Discussion paper, 4*, 144.

[5] Mell, P., & Grance, T. (2011). The NIST Definition of Cloud Computing (Draft) Recommendations of the National Institute of Standards and Technology. *Computer Security Division Information Technology Laboratory National Institute of Standards and Technology. Gaithersburg*, *Januar*, 7.

Studying out the didactic potential of these methods, we found out that the researchers make use of such concepts as *information and communication technology* (ICT) and the *computer-based technologies*. To differentiate these concepts and to specify which of them is to be used in future, we have analyzed the opinions of such scholars as M. Zhaldak[6] (2013) and Y. Trius[7] (2014), whose works are frequently referred to, considering the application of the methods under study in teaching mathematical disciplines.

M. Zhaldak[8] (2013) views information and communication methods as methods and technical means of collecting, storage, organization, transmission, processing, and presentation of data that ensure a better quality of a person's knowledge and develop skills of managing technical and social processes. In Yu. Trius'[9] (2014) view, information and communication methods are the methods employing the informatization tools with the computer technology occupying a leading place, and its application, at all stages of training within a certain discipline, makes it possible to use computer-based training technology as a systematic support of the process of transferring and learning of students' knowledge and skills. Consequently, ICT is a more unified technology, the use of which involves the integration of telecommunication and computers, programs and software, and involvement of cumulative and audiovisual systems. ICT contains information technologies, computer-oriented technologies, telecommunications, media broadcast, all types of audio and

[6] Zhaldak, M. (2013). Computer use in the educational process should be pedagogically balanced. *Informaty`ka ta informacijni texnologiyi v navchal`ny`x zakladax, 1*, 10-18. (in Ukr.).

[7] Tryus, Y., & Kachala, T. (2014). Cloud technologies in management and educational process of Ukrainian technical universities. *Informacijni texnologiyi v osviti, 19,* 22-33.

[8] Zhaldak, M. (2013). Computer use in the educational process should be pedagogically balanced. *Informaty`ka ta informacijni texnologiyi v navchal`ny`x zakladax, 1*, 10-18. (in Ukr.). Zhaldak, M. (2013). Computer use in the educational process should be pedagogically balanced. *Informaty`ka ta informacijni texnologiyi v navchal`ny`x zakladax, 1*, 10-18. (in Ukr.).

[9] Tryus, Y., & Kachala, T. (2014). Cloud technologies in management and educational process of Ukrainian technical universities. *Informacijni texnologiyi v osviti, 19,* 22-33.

video processing and transmission, network management, and monitoring.

Taking into account the facts mentioned above, it has been claimed that it is erroneous to equate information and communication technology and computer-based technology.

Consequently, it is not correct, while describing the use of specific ICT tools (presentations, systems of computer mathematics (SCM), and electronic means) to use a general concept of ICT. Taking into consideration the fact that in the process of mastering DE by bachelor students in information technologies we do not use all the potential of ICT in the training process, the notion of computer-based training methods will be applied.

The analysis of scientific studies by K. Vlasenko, N. Rotan`ova & I. Sitak[10] (2016), D. Gubar[11] (2013), V. Klochko & Z. Bondarenko[12] (2010) and other scholars in the field of application of computer-based methods in teaching mathematical disciplines to IT students, enables defining the main didactic opportunities of these technologies, namely:

- implementation of rational organization of educational and professional activities of students through the appropriate structuring of training material and its rearrangement according to a certain scheme or procedure;
- engagement of all kinds of sensory perception of students while training by means of synergistic combination of visual and auditory perception;
- ensuring the transparency of training exercises, which provides individualization of students' educational process and building their own learning trajectory while maintaining autonomy of their cognition;

[10] Vlasenko, K., Rotan`ova, N., & Sitak, I. (2016). The design of the components of a computer-oriented metodical system of teaching differential equations to future information technology specialists. *International Journal of Engineering Research and Development, 12(12)*, 09-16.

[11] Gubar, D. (2013). *Methodology of creation and application of interactive tools for teaching analytical geometry to classical university students*. Donetsk, Ukraine : Donetsk National University. (in Ukr.).

[12] Klochko, V., & Bondarenko, Z. (2010). *Generating IT students' understanding of solving differential equations*. Vinnitsa, Ukraine : VNTU. (in Ukr).

- use of simulated problem situations in the process of familiarizing students with the skills necessary for the future professional activity of a specialist, by the use of versatile presentation of the essence of the training material and computer simulation.

Let us consider whether it is necessary to involve these options in the process of teaching DE to bachelor students in information technologies.

A stable interdependence and the continuity of the instructional material are within the peculiarities of the content presentation of the discipline "Differential Equations", as well as that of any other mathematical discipline. For example, some equations of second order through some substitution are reduced to the equations of first order, and most differential equations of first order, in their turn, can be reduced to equations with separable variables by means of certain transformations. Consequently, when mastering the solution of differential equations, a student has to constantly turn to the previously acquired knowledge and skills. With a small amount of time, allocated for the discipline, and a significant amount of instructional material, it is very important for the IT students to get an idea of a general course structure and learn how to quickly navigate the training messages with the purpose of effective mastering DE. For this purpose, the instructional material should be expressly structured. It is also necessary to group theoretical information and examples of problem solving according to certain types and attributes. This goal can be achieved with the help of computer-based methods. This fact is confirmed by V. Klochko, & Z. Bondarenko[13], who consider that it is possible to achieve structuring theoretical material for the module "Differential equations" by computer-oriented support.

A specified support can provide the perception of the important part of mathematical modeling course by a student. Based on experience, sensory perception during

[13] Klochko, V., & Bondarenko, Z. (2010). *Generating IT students' understanding of solving differential equations*. Vinnitsa, Ukraine : VNTU. (in Ukr).

the construction of differential models usually causes difficulties for students because it requires a lot of extra “non-mathematical” knowledge of physical or chemical condition of the object of study, or physical, economic, biological and other laws and processes. Computer-based teaching methods enable to create multi-sensory interactive learning environment that may be accompanied by conscious perception and understanding of the process under study, and the implementation of its mathematical description.

Applying the respective procedures of solving equations of various types generates the ability to solve differential models. It is important to repeatedly implement each procedure to generate the proper actions of the students. In I. Mikhailenko’s[14] (2016) view, computer-based methods may accompany the process of doing training exercises, taking into consideration the differentiations of training and individual abilities. Mikhailenko’s conclusion is consistent with the recommendations of D. Gubar[15] (2013), who indicates the relevance of bringing these technologies into reflecting individual characteristics and capabilities of students’ assimilation of the initial reports. Each of the bachelor students spends a different amount of time for mastering certain instructional material; each student chooses their individual right time to study the discipline (asynchronous learning).

Providing both synchronous and asynchronous modes of learning, the method under study can help a teacher to organize training and professional activity of bachelor students. They must analyze the adequacy of the model of the object of research, adhere to the procedures of solving problems of modeling objects and processes of informatization, problems of optimization, prediction, optimal control and decision-making, develop the concept of computer implementation of the model of the researched subject, and explore the manageability of the models in future professional activities.

[14] Mikhailenko, I. (2016). *Methods of teaching differential equations to future mechanical engineers*. Kharkiv, Ukraine : Kharkiv National University G. S. Skovoroda’s name (in Ukr.).

[15] Gubar, D. (2013). *Methodology of creation and application of interactive tools for teaching analytical geometry to classical university students*. Donetsk, Ukraine : Donetsk National University. (in Ukr.).

S. Rakov[16] (2010) concludes that the generation of the above-mentioned skills should be ensured via computer simulation, thus enabling versatile instructing and with the latter facilitating the analysis of the instructions, which, in its turn, makes possible the interpretation of the relationship between the academic content of the components and the overall structure. Moreover, the methods under study can help an instructor to provide the support to problem situations and manage the students by means of computer simulation that will ensure their awareness of the contents of the learning material and its various semantic and symbolic interpretations.

Thus, the use of computer-based methods is offered as a systematic method to accompany the process of transferring knowledge and skills to students and their assimilation when teaching DE.

The consensus is that *support* is an operation that accompanies a specific action or a process. We examine the process of teaching DE to IT students. The accompaniment of such a process by computer-based method means providing the students with tutorial messages in the demo form employing gadgets (laptop, mobile phone, tablet, projector etc.), and networking resources. Taking into account that the appropriate support is made possible with the help of software, we can consider it to be computer-based and make use of the software tools designed for demonstrational modeling, ensuring the component of the activity-based and subject-oriented environment, determining the students' educational performance, and obtaining referential communications.

The tools enabling demonstrational simulating, can be used for the explanation of the new instructional material, which is accompanied by the demonstration of the model of the object under study. This approach provides a synergistic combination of visual and auditory perceptions of instructional material. The models can be simulation models, simulation-driven models, and dynamically controlled models. The latter type can be based on a mathematical description of a process

[16] Rakov, S. (2010). Problems of information education in Ukraine. *Komp'yuter u shkoli ta sim'yi, 2*, 34-35. (in Ukr.).

that is as close as possible to the scientific models of a particular subject field and therefore is opened to students. Dynamic models, through animation and semi-automatic control, can help a teacher to create the models of objects or processes of cognition, implement the principle of simulation of learning situations in the process of explanation of a training material and organize problem solving.

Video fragments, recorded on digital media, demonstration and reference systems, and other tools can be referred to the tools in question.

Software tools, that enable the visualization of the training objects with the performance over them of certain actions or transformations, are primarily included into the tools, which provide the components of active object-oriented environment.

Pedagogical software programs (PSP) and GRAN DG, developed under the leadership of such Ukrainian scientists as M. Zhaldak[17] (2013) & S. Rakov[18] (2010), are at the forefront of these recourses. Moreover, systems of computer mathematics provide fast and high-quality performance of numerical calculations, analytical transformations, and construction of 2D and 3D graphs. The considerable experience of scientists in using the above-mentioned software tools, while teaching mathematical disciplines, is applied. However, another computer-based support that can be used for organization of training and professional activities of bachelor students is being sought. This should be a support for providing the components of the activity-based and subject-oriented environment and contributing to the formation of skills of students during the procedures of solving differential equations and DE systems. Such tools include computer simulators, used as algorithmic hands-on exercises; their interface allows students to work out the process of studying step-by-step, with parallel correction of errors during the decision-making.

[17] Zhaldak, M. (2013). Computer use in the educational process should be pedagogically balanced. *Informaty`ka ta informacijni texnologiyi v navchal`ny`x zakladax, 1*, 10-18. (in Ukr.).

[18] Rakov, S. (2010). Problems of information education in Ukraine. *Komp'yuter u shkoli ta sim'yi, 2*, 34-35. (in Ukr.).

The use of simulators during mastering actions of solving DE contributes to the gradual automation of actions, loss of its novelty, and the formation of the skills.

Cloud computing, which allows the user an easy access to the calculations through the appropriate resources, can provide the verification of the formation of appropriate actions of students. The speed and easiness of receiving services from such resources have contributed to the rapid increase of the number of studies of cloud technology use in teaching higher mathematics.

Correction of the respective actions of students is also possible through the determination of the level of educational achievements. These tools include tests that differ in the way of placement, structure, and possible ways of submitting an answer, the degree of "flexibility", and the completeness of the coverage of the training material.

Besides, scientists have not fully examined pedagogical software tools used for information purposes. These recourses can be used as a supplement to traditional textbooks. It can be e-manuals, textbooks, handbooks, databases, and knowledge base with a text or a multimedia presentation of educational material, hypertext or hyper-media systems. By way of storage reference materials can be stored based on concentrated or distributed model of data storage. Educational sites are a natural form of such data storage. The educational site is a consistent, reasonable conceptual and structural system that combines interrelated web pages, the content of which is subordinated to the general idea expressed in specific goals and objectives of each of them.

In Gubar's [19] view (2013), the need for improvement of the ways and methods of interaction between subjects of the educational process among themselves, increase of the level of interactivity of the didactic material, and the lack of a single agreed system for the use of interactive learning tools in the process of training specialists, determine the urgency of the development of educational resources. We agree with the researcher that the use of telecommunication networks that

[19] Gubar, D. (2013). *Methodology of creation and application of interactive tools for teaching analytical geometry to classical university students*. Donetsk, Ukraine : Donetsk National University. (in Ukr.).

enables the organization of higher mathematical education through blended learning can ensure the interactivity of teaching tools.

Analyzing different approaches of the definition of the concept of blended learning, we can state that the involvement of this type of training requires from a teacher a combination of traditional and computer-based methods, ensuring social interaction ("student-teacher", "student-student") that is significant for students of the studied specialty.

Let us have a closer look at the requirements for the development of a mixed DE study course for IT students.

The training content of the DE course must be submitted in the state language. It should contain a necessary didactic support of the discipline under study; that is a guide on the course and its program, interactive lectures on each topic, a set of methodical recommendations for practical work, procedures for solving basic types of differential equations, training simulators for working out skills of solving DE, dynamic models for the formation of mathematical modeling skills, online calculators for checking the correctness of calculations, PSP for visualization of teaching objects, SCM for facilitation of the implementation of complex calculations and implementation of numerical methods for computation of differential equations and DE systems, an electronic library of discipline with links to tutorials, educational sites, and a virtual classroom.

Such course can be placed on the site, which is possible with the involvement of Content Management Systems (CMS) WordPress. CMS have software that provides tools for adding, editing, and deleting information on a web site. Most of the modern CMS have a modular architecture that allows the administrator to choose and customize the components that it needs. Similar to the WordPress CMS can manage the text and graphic content of the website, providing the user with the interface for work with the web site content, useful tools for the storage and publication of information by automating the process of placing information in databases and its distribution in the Internet space.

While choosing CMS among the most common ones, we have found out that WordPress has a number of advantages over other systems such as ease of installation, simplicity of settings, ease of administration, high level of functionality, high speed of work, flexible seo-optimization capability, and support for modern web standards. Moreover, WordPress is free. This is due to the use of the PHP programming language using the MySQL database and the distribution of source code on the terms of the GNU General Public License.

Besides, the experience of using WordPress[20] (Williams, Richard, & Tadlock, 2011) in an educational sphere, since 2008, has testified its reliability and convenience. WordPress has become a powerful tool for creating websites of Newspapers, magazines, universities and colleges. It has been used as a control system of educational content and learning process. We have analyzed the content of existing sites, which can provide a mixed training DE to bachelor students (Table 3.2.1). This analysis has shown that existing developments do not meet the above requirements.

Most of the considered network resources include training materials for mastering DE of first and second order, and sometimes DE systems. There is almost no attention to the issues of mathematical modeling with the use of differential equations and the theory of stability on sites. The lack of computer simulators, which should help develop skills of students in usage of procedures of solving differential equations, has been paid attention to. Moreover, the sites almost do not have the facilities for demonstration modeling, providing all components of the activity of subject-oriented environment and determining the level of academic performance. In addition, computer-based tools that have a reference assignment do not meet the requirements of the educational program of bachelor students.

Let us specify which of the computer-based tools can be used by teachers at different stages of the formation of actions in the process of training DE of bachelor students in information technologies (Table 3.2.2).

[20] Williams, B., Richard, O., & Tadlock, J. (2011). *Professional WordPress Plugin Development*. Canada: Wiley Publishing.

With the application of the tools mentioned above along with the corresponding methods and forms of studying, the support of mastering certain abilities by bachelor students is provided. The formation of readiness of future specialists to use ICT in their professional activities occupies a significant place. In Morze's[21] (2013) opinion, mastering skills of ICT application in practical activity, contributes to the formation of computer literacy of students and ICT literacy. The formation of ICT literacy of bachelor students is a prerequisite for the development of their IT competence, which is the main one among professional skills.

In this study it has been showcased that the use of computer-based methods in the course of teaching differential equations to the IT bachelor students is based on the creation and application of an educational site, during the development and involvement of which it is important to provide the choice of the appropriate components of the methodological system.

[21] Morze, N., Kuzminska, O., & Protsenko, G. (2013). Public Information Environment of a Modern University, ICT in Education, Research and Industrial Applications: Integration, Harmonization and Knowledge Transfer. *CEUR Workshop Proceedings, 1000*, 264-272.

The title of the site	Site content
1	2
Ukrainian Internet Resources	
AlWEBRa Educational Channel	Contains educational materials to the topics “Ordinary differential equations” and “Differential equations of higher orders”: - theoretical materials; - video fragments of lectures; - training tests
“YukhymCommunity” Educational portal of mathematics	Contains examples of solving basic types of differential equations of first and second order
Russian Internet Resources	
“Differential Equations»	Contains teaching materials to the topics “Basic concepts of the theory of differential equations”, “Ordinary DE of first order”, “Existence and unity of solving DE”, “Ordinary DE of higher order”, “Theory of linear DE”, “Systems of ordinary DE”, “Theory of stability”, “Partial differential Equations of first order”: - theoretical materials; - tasks for independent work
Internet Resource “EXPonenta.ru»	Contains educational and reference materials to the topics “DE of first order”, “DE of higher order”, “Systems of ordinary DE”, “Autonomous systems of DE”, “Stationary points of DE systems”: - theoretical materials; - examples of analytical solutions of DE; - Examples of solving DE using such systems of computer mathematics as MathCad and Mathematica; - tasks for self-mastering the topic; - control questions
“ma∑prof∫.ru»	Contains teaching materials to the topics “DE of first order, “DE of second order”, “Systems of ordinary DE”: - theoretical materials; - numerical methods of solving DE: Ejler’s and Runge-Kutt’s methods; - examples of some geometric and chemical problems that can be solved with the use of DE

1	2
English-language Internet resources	
“Interactive Mathematics»	Contains educational and reference materials to such topics as “DE of first order”, “DE of higher order”: - theoretical materials; - examples of solving problems to each topic; - some examples of DE use in the problems of biology, physics, and mechanics
“Interactive Differential Equations»	Contains educational materials to such topics as “DE of first order”, “ DE of second order”, “systems of ordinary DE”: - theoretical materials; - reference data; - glossary
Advances in Difference Equations	Contains: - scientific articles on the topics of the use of DE; - examples of solutions of typical DE of first order; - e-library on the subject
MathWorks	Contains training materials to the topics “DE of first order” and “DE of higher order”: - video lectures MatLab Central by Prof. G. Strang and K. Mole - examples of the use of MatLab SCM for solving DE of first and second order
“Notes on Diffy Qs: Differential Equations for Engineers»	Contains teaching materials to the topics “DE of first order”, “DE allowed for the reduction of order: - interactive lectures; - examples of analytical solving of DE - examples of implementation of the numerical solution of DE by SCM MathLab tools; - graphical interpretation of solutions of equations; - illustrations of physical processes for the description of which DE are used

Table 3.2.1 Available Internet resources for training DE.

Name of a stage	Stage purpose	Computer-oriented tools, which must accompany the stages of the action formation
1	2	3
The formation of materialized actions	- mastering the mathematical subject knowledge: DE of first order, DE of higher order, linear DE with constant coefficients, finding the approximate solutions of the Cauchy problem, DE systems, the theory of stability of solutions of DE, and their systems; - the development of academic skills: application of procedures of solving different types of first order DE, higher order DE and systems of DE ; - the use of procedures of involving software tools in solving tasks for the development and research of algorithms for the functioning of computerized systems by methods of integrating linear DE order	- Interactive lectures; - Interactive recommendations for practical classes - Procedures of solving DE of the corresponding types; - computer simulators; - online calculators; - instructions for use of software tools in the process of solving tasks
Formation of speech actions	- recognition of types of DE of first and higher orders; - compliance of stages of analysis of differential models for correctness, completeness, complexity and accuracy; - recognition of the stages of the study of existence, uniqueness and stability of solutions on the basis of the theory of stability of DE solutions	- interactive lectures; - interactive recommendations for practical classes; - interactive recommendations for practical classes; - procedures for solving DE of the corresponding types; - test assignments; - dynamic models; - virtual classroom

1	2	3
Formation of mental actions	- formation of mathematical modeling skills with the help of DE of first order, linear DE with constant coefficients, and normal DE systems; - familiarization with the skills which are necessary for future professional activity, analysis of the adequacy of the model of a studied subject, observance of procedures of solving problems of modeling objects and processes of information, optimization problems, forecasting, optimal control and decision making, development of the concept of computer model implementation of a studied subject, study of controllability of models	- dynamic models; - online calculators; - PSP for visualization of objects of study; - SCM Maxima Scilab; - virtual classroom

Table 3.2.2 Computer-oriented tools, which should accompany the stages of formation of IT Bachelor students' activities during training DE.

3.3. Teaching Functional Analysis in a Pedagogical University: a Hands-on Course

Iryna Lovianova, Dmytro Bobyliev

Stating the problem

The subject of the course "Functional Analysis" is the scope of functions and their reflection. Functional analysis as an independent section of mathematics developed at the beginning of the last century as a result of the generalization of mathematical analysis structures, linear algebra and geometry. Since then, its ideas and methods have penetrated into all the fields of mathematics, physics and applied sciences on the rights of a powerful generalization theory and a convenient tool for the specific problems study.

The study of functional analysis is typical for mathematical specialties of classical universities. Nevertheless, in the pedagogical higher educational institutions this course is found in the curricula of the specialty 014.04 Secondary education (Mathematics) with the additional specialty 014.09 Secondary education (Computer Science). The bachelor's program in these specialties involves studying the elements of functional analysis. It is usually the basic part of the fundamental cycle with the academic study time one semester, while the number of class hours is small.

The framework of academic study duration, its applied orientation and the students' level of basic training in modern pedagogical universities do not allow them to learn such a complex mathematical discipline from the standpoint of the classical approach, which provides for the fundamental and self-sufficient supply of purely theoretical material. In addition, pragmatically-minded students are not interested in the idea of generalization and formalization of mathematical constructions. Obviously, the motivation increases if you bring the academic course to computing practice with compulsory engagement of computer technology. For future mathematics and computer science teachers it is necessary to emphasize the applied role of functional analysis, which is reduced to the analytical substantiation of the effectiveness of the numerical methods application.

Research and publications analysis

There is only insufficient part of those who put and solve a similar methodological problem among all the published educational resources. For example, the textbooks by V. Trenogin, B. Pysarevskyi & T. Soboleva[1] (1984), V. Lebedev[2] (2005) are oriented on applied specialties. However, they are too voluminous, complex (especially V. Trenogin et al., (1984) and, although they cover functional analysis from the point of view of numerical methods, are not completely the path from the idea to the calculation formula, which complicates their use in the pedagogical university. Moreover, essential adaptation of these textbooks is required.

In the problem books of functional analysis it is not accepted to accentuate either the computational or even algorithmic component, and they are not accustomed to actively involve computer technologies. The existing sets are dominated by theoretical tasks. They tend to use purely abstract schemes (space X, norm p, operator A, etc.). The overwhelming majority of such tasks are antipodes of typical calculations. The embodiment of this approach is a problem book by O. Kirilov, & O. Gvishiani[3] (1988), recommended for classical universities, however, to this or that degree, all the problem books of functional analysis tend to immerse into the formal logic apparatus. When setting the training problem for future mathematics and computer science teachers, at least, it would be necessary to change the form of the presentation of traditional tasks: to move from abstract to specific spaces, norms, operators, and from

[1] Trenogin, V. A., Pisarevsky, V. M., & Soboleva, T. S. (1984). *Problems and exercises – functional analysis*. Moscow, USSR: Nauka. (in Rus).

[2] Lebedev, V. I. (2005). *Functional Analysis and Computational Mathematics* (4th ed.). Moscow, Russia: Fizmatlit. (in Rus).

[3] Kirillov, A. A, & Gvishiani, A. D. (1988). *Theorems and problems of functional analysis* (2nd ed.). Moscow, USSR: Nauka. (in Rus).

tasks to bring to more typical, algorithmic calculations and constructs. This way they are realized in the workshop of Belarus State University[4] (Antonevich et al., 2003), though not to the full extend: in the range of tasks, there are no simple enough one-hour exercises aimed at working out the elementary instruments of the discipline.

Another way of bringing functional analysis to computational practice is to put in the center of the problem the application of any numerical method, followed from the theory. This tendency is revealed episodically in the problem books by V. Trenogin, B. Pysarevskyi, & T. Soboleva[5] (1984). Based on the textbook[6] (Trenogin, 2007), this task allows us to develop the application of functional analysis to numerical methods thoroughly, but, unfortunately, it is carried out mostly from the theoretical side, not practical, and at a high level of abstraction. In addition, the application of numerical methods requires the use of computer technology, and such attempts are found in A. Antonevich, et al.,[7] (2003) & V. Trenogin, but their weight is negligible. One can conclude that in the existing collections functional analysis problem books, there are almost no problems with the application of functional analysis to numerical methods that would be useful for the future mathematics and computer science teacher, and that would allow both mathematical argumentation and implementation with the use of computing.

The objective of the article

To substantiate scientifically the expediency of the developed educational-methodical complex on functional analysis,

[4] Antonevich, A. B., Vatkina, E. I., Mazel, M. A., Mirotin, A. R., Mukhin, V. V, Radyno, Ya. V. et al. (2003). *Functional Analysis and Integral Equations: Laboratory Practice*. Minsk, Belarus: BSU. (in Rus).

[5] Trenogin, V. A., Pisarevsky, V. M., & Soboleva, T. S. (1984). *Problems and exercises – functional analysis*. Moscow, USSR: Nauka. (in Rus).

[6] Trenogin, V. A. (2007). *Functional analysis*. Moscow, Russia: Fizmatlit. (in Rus).

[7] Antonevich, A. B., Vatkina, E. I., Mazel, M. A., Mirotin, A. R., Mukhin, V. V, Radyno, Ya. V. et al. (2003). *Functional Analysis and Integral Equations: Laboratory Practice*. Minsk, Belarus: BSU. (in Rus).

directed on formation of the future mathematics and computer science teacher's general and professional competences.

Presenting the main material

Insufficient attention to the practice of applying numerical methods in the existing functional analysis manuals compiled for applied specialties can be explained by several circumstances. Firstly, the very ideology of functional analysis is tuned to the high abstractness of this section of mathematics. Secondly, the training trajectories of this discipline were structured at a time when computer technologies were still far from the leading role in education, and therefore, their connection to the educational process was not perceived as something natural and not burdensome. Thirdly, numerical methods are traditionally presented in a separate course of computational mathematics (or course of numerical methods). But A. Myshkis[8] (2003) noted that in terms of technical university "it is dangerous to allocate all computational issues to a separate section of the mathematics course: such a separation can significantly reduce the idea of algorithmicity in other sections of the course, which appear to be opposed to the calculations and thus blurred in the applied relation". This view is also relevant for the training of future mathematics and computer science teachers in pedagogical universities. Let's add to this argument another, due to the current state of education: it makes no sense to break the justification of the method and its first trial application. The approximation of the course of functional analysis to computational mathematics contributes to the continuity and coherence of vocational training. Perhaps this is even the only way to fully implement functional analysis in a pedagogical university. The convergence with computational mathematics should be such as to fully prove the theoretical fact to the number: to trace the projection of abstract ideas into the plane of numerical methods and to give an opportunity to immediately test methods in computational practice.

[8] Myshkis, A. D. (2003) On teaching mathematics to the applied sciences students. *Matematika v vysshem obrazovanii, 1,* 37-52. (in Rus).

Of course, the measure of this convergence should be reasonable, so that functional analysis does not lose its identity and is not substituted by the course of computational mathematics.

To solve these problems, a scientific methodological research was conducted and a set of two textbooks was developed: a summary of lectures[9] (Bobyliev, 2016) and a collection of tasks[10] (Bobyliev, 2017) on functional analysis for pedagogical universities. The basic concepts underlying this project are as follows:

- adaptation of training material to the students' level of preparation and analytical skills;
- cultivation of the applied component of the discipline, which is realized by a combination of functional analysis and computational mathematics;
- modernization of the course for the use of computing means (applied mathematical packages).

Let's consider in more detail how these concepts are implemented in the textbooks (Bobyliev, 2016[11], 2017[12]).

The set of lectures[11] contains short theoretical information about the basic modules of functional analysis: the theory of compression operators, the theory of Fourier series in the Hilbert space and the theory of linear operators. Moreover, some complex structures with purely academic values have been disregarded, in particular, those not having a visual application in computational practice.

For example, many elements of the topology and all the adjacent theorems, the concept of conjugate space and operator, the Banach theorem on inverse operators, the theorem on the addition of a metric space (considered at the level of formulation), theorems on the extension of the operator, the functional, and others are excluded.

[9] Bobyliev, D. Ye. (2016). *Functional analysis*. Kriviy Rig, Ukraine: Dionat. (in Ukr).

[10] Bobyliev, D. Ye. (2017). *Functional analysis: collections of tasks*. Kriviy Rig, Ukraine: Dionat. (in Ukr).

[11] Bobyliev, D. Ye. (2016). *Functional analysis*. Kriviy Rig, Ukraine: Dionat. (in Ukr)

[12] Bobyliev, D. Ye. (2017). *Functional analysis: collections of tasks*. Kriviy Rig, Ukraine: Dionat. (in Ukr)

The notion of a linear operator limitation, a key to most textbooks on functional analysis, is replaced by the concept of continuity which is equivalent to it. The reason for the replacement is that the traditional definition of a limited linear operator does not correspond to the definition of a bounded function taken in the course of mathematical analysis, whereas the universal concept of continuity for an operator in metric spaces is consistent with the continuity of a function from the course of mathematical analysis. The continuity of the operator is defined as the ability to maintain the sequence convergence, since this definition is the easiest and is used in numerical methods.

The course deals only with the description of the Lebesgue integral included for the introduction of the Lebesgue spaces, and is constructed without the support of the theory of measure, since this section of the bachelor's course in mathematical analysis in most pedagogical universities is not taught to future mathematics and computer science teachers, and it cannot be considered in a short course of functional analysis.

The description of the functional analysis is oriented on two basic tasks that are at the intersection of fundamental and applied mathematics: the approximation of functions and the solution of operator equations. In the problems of approximation of functions with orthogonal systems, the questions of accuracy and quality of approximation are raised. When solving operator equations, the issue of convergence of numerical methods and control of the accuracy of the approximate solution are highlighted on the foreground. The problem of uniqueness of the solution, too, does not fall out from the field of consideration. But the problem of existence, rather difficult for comprehension, is set aside and in some places omitted.

In each module of the lecture set, the line of presentation of theoretical material passes from the introduction of basic concepts to the proof of key theorems that have direct access to widely known numerical methods. These "outputs" are usually described in the last paragraphs of the modules. For their compilation, a database of several dozen textbooks in both functional analysis and different sections of computational mathematics was currently analyzed.

The first module is devoted to metric spaces and compression operators. It ends with an overview of the problems in which the application of the principle of compression operators and the method of simple iterations can be used for the approximate solution of equations of different types.

The second module represents the theory of Fourier series in the Hilbert space. Considerable attention is paid to the variety of orthogonal systems: trigonometric, polynomial and systems of step functions. The module completes the description of the Fourier series connection with the problem of approximation and the explanation of such essential features as the character of the convergence of the Fourier series, the specificity of trigonometric and polynomial approximations, and the differences between the Fourier and Taylor series.

The third module is devoted to the theory of linear operators and covers related issues of a functional optimization. The statement ends with a description of the variational and projective approach to the approximate solution of linear operator equations. The method of least squares and the Galerkin method are analyzed in detail. In addition, in the third module, there are other outcomes for computational mathematics, presented in various sections: search for the solution of an equation in the form of a Fourier series by its own functions of the operator, solving the integral equation by the method of replacing the nucleus into a degenerate, approximate minimization of the functional by the Ritz method.

In addition to the strictly deductive method of teaching, heuristic methods (using analogy, selective verification) and other learning methods are used, including those based on informal ideas of mathematical constructions. Most paragraphs have a preface outlining the relationship between the proposed mathematical constructs of functional analysis with similar constructs of the mathematical one, geometry or linear algebra, and some of the brightest applications in natural science and technology.

Thus, the course does not claim to reveal the completeness of the theory, but focuses on the algorithmic component of the functional analysis. The presentation of theoretical constructions is simplified to elementary when trying to preserve the classical quality of this discipline and not to miss the idea that is laid down in this section of mathematics. The abstract energy of the functional analysis is substantially curtailed in comparison with the academic course, but is sustained in such a volume that it could reasonably be applied.

The set of lectures and problems are constructed in such a way as to make the subject as accessible as possible for self-study and, from the first lessons, to intensify the student's educational activity in the context of strict curriculum restrictions. If you do not spend lessons on a consistent and detailed presentation of the material, and adapt the study process to the mode of review lectures and tutorials with the problems solving, then it makes possible to optimize the amount of class hours. It is expedient to limit review lectures to a small amount of formal data and to devote a comprehensive discussion of key mathematical ideas of functional analysis and related non-mathematical associations. The course of functional analysis urgently needs this approach, since it brings a certain summary of the accumulated experience in the study of other sections of mathematics.

The review lectures can include, for example, the following issues: the problem of similarity and differences between objects that are numerically solved using different metrics; what is the convergence and the reasons for its diversity; Linearity as a universal positive characteristic of mathematical constructions and nonlinearity as a source of troubles; what is the difference between a schedule and an approximation of functions; the general notion of continuity and the role played by continuity in various numerical methods; relative simplicity and attractiveness of finite-dimensional spaces, numerous methods based on reduction to a finite-dimensional problem. The set of lectures is intended to prepare and inspire the student to reflect on the ideology that lies in the course of functional analysis. Certain relevant review lectures can greatly enhance this effect.

The problem book contains a large bank of multilevel tasks with a large number of options and is designed for a convenient distribution of points in the evaluation: for example, from 1 point for the simplest one-step problem to 10 points for multi-step calculation with implementation in a mathematical package.

Consider examples of problems from the collection.

Exercise 1. Prove directly that if E is a Banach space and M is a closed subspace of E then the quotient space (E / M, $\|\cdot\|_{E/M}$) is complete. [Use Banach's criterion (Lax[13], 2002)].

Exercise 2. Let M and N be subspaces of the normed space X. Prove that if M is finite dimensional and N is closed then $M + N$ is closed. [Recall that finite-dimensional subspaces of normed spaces are closed; use the quotient map].

Exercise 3. Find a Hilbert space H and a countable family of vectors (x_n), $n \in N$ in H that is summable but not absolutely summable (i.e., $(\|x_n\|)$, $n \in N$ is not summable) (Vincent-Smith[14], 1991).

Exercise 4. A sequence in a normed vector space that is convergent is necessarily bounded. Is the same true for nets?

Exercise 5. Prove that the closed unit ball of c_0 has no extreme points.

Exercise 6. Let H be a Hilbert space. Prove that every unit vector in H is an extreme point of the closed unit ball H_1. [Note that 1 is an extreme point of F_1]. Deduce that every isometry in $B(H)$ is an extreme point of the closed unit ball $B(H)_1$.

Exercise 7. Let X be a Hausdorff, locally compact space. Prove that $C_0(X)$, the unitization of the algebra of continuous functions on X that vanish at infinity, is topologically isomorphic to $C(X)$, the algebra of continuous functions on X, the one-point compactification of X.

Exercise 8. Let $A = C[z]$ denote the unital algebra of complex polynomials and let $\|p\| := \sup\{|p(\alpha)|: |\alpha| \leq 1\}$ for all $p \in A$. Show that $(A, \|\cdot\|)$ is a unital, normed algebra which is not complete.

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[13] Lax, P. (2002). *Functional analysis*. New York, USA: Wiley & Sons, Inc.

[14] Vincent-Smith, G. F. (1991). *B4: Analysis. Mathematical Institute notes*. Oxford, UK: University of Oxford.

Exercise 9. Let A be a (non-unital) Banach algebra such that every element is nilpotent (i. e., for all $a \in A$ there exists $n \in \boldsymbol{N}$ such that $a^n = 0$). Prove that A is uniformly nilpotent: there exists $N \in \boldsymbol{N}$ such that $a^N = 0$ for all $a \in$ $\in A$. [Consider the decomposition].

Exercise 10. Let A be a unital Banach algebra over **C** and let

$e^a := \sum_{n=0}^{\infty} \frac{a^n}{n!}$ for all $a \in A$. Prove that $e^{a+b} = e^a e^b$ if a and b commute. Deduce that e^a is invertible. Prove further that $f\colon \lambda \to e^{\lambda a}$ is holomorphic everywhere, with $f'(\lambda) =$ $= af(\lambda) = f(\lambda)a$, for all $a \in A$.

When conducting practical classes on the basis of this teaching-methodical complex, the emphasis is transferred from the manual calculation technique to the organization of the computational process on computers. Such a transfer is inevitable in modern conditions.

Conclusions

The considered educational-methodical complex (Bobyliev, 2016[15], 2017[16]) allows and partially forces to reorganize the educational process of teaching functional analysis in a pedagogical university.

It is expedient to use computers and the qualified interpretation of results should become one of the main goals of teaching not only functional analysis, but, mathematics in general at a pedagogical university. The developed problem book makes it possible; on the one hand, to replicate previously acquired skills in solving various problems, on the other hand, it allows students to learn to use mathematical packages. The author's problem book includes 58 tasks (20 variants of each), all of them have samples or instructions to solutions and a corresponding reference to the set of author's lectures.

[15] Bobyliev, D. Ye. (2016). *Functional analysis*. Kriviy Rig, Ukraine: Dionat. (in Ukr).

[16] Bobyliev, D. Ye. (2017). *Functional analysis: collections of tasks*. Kriviy Rig, Ukraine: Dionat. (in Ukr).

Prospects for further research in the field

In the author's teaching-methodological complex certain specific problems related to the differences of scholastic (educational, theoretical) and computer mathematics are not worked out thoroughly, they require attention at the initial stage of the mathematical packages application. The number of tasks in the second problem book showing the typical difficulties that students face when the computer responds in the form of a character expression that may contain special functions, faced by the student for the first time, is insufficient. Applied mathematical packages require a much more responsible attitude to working with data types (numbers, variables, expressions, functions) than it is customary in fast calculations on paper.

3.4. Mathematical and Cultural Potential of the Course "Theory of Measure and Integral" and Its Applications

Mykola Tretyak

During the last decade, the issue of fostering the University students' mathematical culture (MC) attracts the scholars' attention, which is reflected in numerous publications. Most of them focus on the improvement of building pedagogical environment and search for the instruments of the MC formation[1] (Gnedenko, 2006), the aspects of building the MC of students with different professional specialization, namely engineers, mathematics teachers, primary school teachers, educators, economists, classical scholars, etc. (Lodatko[2], 2011). It seems reasonable, that the formation and development of students' MC takes place within learning different mathematical disciplines. It necessitates the study of mathematical and cultural potential of these disciplines. In this paper we consider the mathematical and cultural opportunities of the course "Theory of measure and integral" (TMaI) in the context of formation and development of mathematics students' MC in classical and pedagogical universities.

Addressing the issue of mathematical education of mathematics students, the reputable mathematician and educator B. V. Gnedenko[3] (2006) pointed out that "Our pedagogical duty is to show the full range of possibilities of the mathematics activity theoretical investigations in mathematics itself, application of mathematics in engineering, economics, natural sciences, organization of manufacturing, in psychology, sociology, research on the history of mathematics and its philosophy, and participation in solving mathematics

[1] Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millennium*. Moscow, Russia: ComKniga. (In Rus).

[2] Lodatko, Ye. O. (2011). *Mathematical culture of the primary school teacher*. Rivne-Slovyansk, Ukraine: Entrepreneur B. I. Motorin. (In Ukr.).

[3] Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millennium*. Moscow, Russia: ComKniga. (In Rus).

education problems at all levels. It is of importance to constantly emphasize that in each area of mathematics listed above there are a lot of challenges which require dealing with. Our duty is to educate the generation prepared to difficult and strict examination arranged by life itself with its constantly changing settings. We must help them and foresee what will be necessary for their future work".

In order to objectively evaluate mathematical and cultural potential of the course TMaI, the brief characteristics of the notion of MC of mathematics students will be considered[4] (Tretyak, 2013).

MC of mathematics students is ***the dynamic integral characteristics of their personality, which in full measure capture the level of the acquirement of the achievements of humanity in the field of mathematics and the ability to adequately perceive the mathematical components of the scientific picture of the world and to build their own educational, professional and social activity in accordance with this perception, to create personal moral, ethical, and aesthetic ideals.***

MC of mathematics students is introduced as a complex dynamic system of mutually dependent and mutually specified characteristics of their personality, which are the elements of MC.

To this we refer the following:

1) mathematical knowledge;
2) mathematical skills and abilities;
3) abstract thinking;
4) formal and logical thinking;
5) functional thinking;
6) probabilistic thinking;
7) algorithmic thinking;
8) mathematical imagination, spatial imagination;
9) using mathematical language and symbols;
10) knowledge and understanding of methodology of mathematics;
11) knowledge and understanding of the key components of mathematics, relationships, promising avenues of mathematics development;

[4] Tretyak, M. V. (2013). Mathematical and cultural potential of the course «Theory of measure and integral». *Science and Education a New Dimension: Pedagogy and Psychology, Vol.5*. (In Ukr.).

12) knowledge and understanding of place and role of mathematics in the system of sciences, application of mathematics;

13) knowledge and understanding of the role of science in the life of an individual or mankind as a whole;

14) knowledge and understanding of how to separate mathematical components, build mathematical models, study them and interpret the results;

15) knowledge of the genesis of mathematics and its main branches, biographies of famous mathematicians;

16) knowledge and understanding of the mathematics and art (music, painting, sculpture, architecture etc.) intercourse;

17) knowledge of mathematical folklore, different illustrative examples, funny stories;

18) possession of well-developed mathematical aesthetical ideals;

19) having well-developed moral and ethical ideals;

20) ability of perception and creative acquisition of new ideas, creativity.

Elements of MC are connected by integration lines I – V:

I – that of knowledge (line of knowledge);

II – that of skills (line of skills and abilities);

III – that of reasoning (line of thinking);

IV – linguistic and symbolic (line of language and symbols);

V – evaluative and reflexive (axiological line). Integration lines are totally natural and necessary details of the structure of MC, they involve all the elements of MC, combining them into one whole.

The following diagram (Figure 3.4.1) illustrates the model of mathematics students’ MC we come up with.

Over the past decades TMaI has become an academic discipline with well-established goals, subject, methods, content, level of rigor, degree of abstraction, conception and methodology of learning. We will analyze mathematical and cultural possibilities of the TMaI course taking into account the ideas and model of MC which were mentioned above. We will consecutively consider those elements of the MC that are most susceptible to the development and enrichment within the study of TMaI.

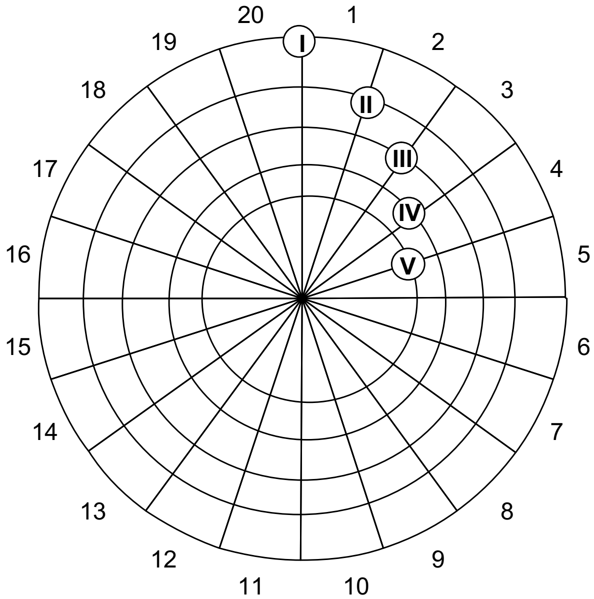

Figure 3.4.1 Schematic representation of MC of student of mathematical specialty.

1. *Mathematical knowledge*. In the study of TMaI this element of MC is widened by the whole set of very important notions (ring, semi-ring, σ-algebra of sets, Borelean's σ-algebra, measure, charge, measurable space, measurable function, absolutely continuous function, singular function, almost everywhere convergence, convergence in measure, convergence in mean, Lebesgue's abstract integral, product of measures, measure and integral of Lebesgue-Stieltjes, Lebesgue spaces) and facts (Carathйodory's theorem, theorems related to different types of convergence of functional sequences, σ-additivity and absolute continuity of integral, theorems about boundary transition under the sign of integral, theorem of Radon–Nikodym, Fubini's theorem). Without this knowledge, higher mathematical education, and hence the MC of mathematics student, cannot be considered a complete system. Moreover, without notions and facts mentioned above, modern learning of functional analysis, probability theory, mathematical statistics, random processes, variations calculus and differential equations with partial derivatives couldn't be possible.

2. Mathematical skills and abilities. This element of MC includes the following important skills and abilities:

a) to carry out union and intersection of sets;
b) to build new measures based on known measures; to build the measures with planned characteristics;
c) to search the limits of functional sequences for different kinds of boundary transitions;
d) to integrate the functions by different measures, in particular, by different measures of Lebesgue-Stieltjes;
e) to find the Hahn’s decomposition, Jordan’s decomposition of charge and Lebesgue’s decomposition of function;
f) to find the Radon–Nikodym’s derivatives;
g) to reduce the multiple integrals to iterated ones by using the Fubini’s theorem;
h) to carry out the change of variables in abstract and classical Lebesgoue integrals;
i) to create new functions using all known methods of functions creation (arithmetical operations, composition, narrowing, extension, presentation by parts, transition to inverted function, boundary transition, parametrical and implicit presentation, parametrical integrals etc.);
j) to work with reflections, for which initial domain or final domain isn’t point sets.

3, 4. Abstract thinking, formal and logical thinking. These elements of MC in the study of TMaI acquire unique opportunities for their development as the main objects considered in TMaI, (measure, measuring function, and integral) in their simplest versions, have already been mastered by students and their generalizations are so wide and all-encompassing that they become possible only under conditions of a high level of abstraction and logical formalism, which requires and encourages the development of abstract thinking and formal and logical thinking. In such a case, studying the proofs of a large number of theorems and solving a large number of tasks serves as a good training for the types of reasoning mentioned above.

5. Functional thinking. Concerning this element of MC, it should be noted that it is difficult to find the academic subject which facilitates functional thinking more than TMaI. It could be explained, in particular, by the fact that in the course of TMaI it is viable to consider many new types of functions and use a wide array of methods of function creations, they relate to a variety of functions whose definition areas and values are both point sets and systems of sets.

8. Mathematical imagination, spatial imagination. It is the matter of common knowledge that the mechanisms of creative mathematical imagination are not well studied. However, under a masterful methodological guidance and application of the didactically suitable mathematical material, the enhancing mathematical imagination could be effective. To make learning efficient, TMaI, as well as any other highly abstract, sufficiently formalized learning discipline, on the one hand, requires a previously developed mathematical imagination, and on the other hand, provides a unique mathematical material and means for its development. Problems on building the measures with planned characteristics, non-measurable sets, non-measurable functions, functions with a given set of points of break, functional sets with planned character of convergence, functions with given integral and differential properties etc., are the examples of such mathematical material.

9. Use of mathematical language and symbols. This element of MC gets facilitating impulses for its enhancing. Firstly, it is the widening of personal apparatus of mathematical terminology and symbols. Secondly, diversification of the ways of formulation and proving of mathematical statements. Thirdly, a wide usage of terminological apparatus and symbols of the theory of sets and mathematical logics. Fourthly, the use of function notions without arguments within the formulation of definitions and theorems, proving theorems and problem solving. For example f, g,... – functions; μ, ν,... – measures, charges; $\int_E f\,d\mu$ – integral; f' – derivative; $\partial_i f$ – partial derivative by *i-variable*; (f_τ), $t \in T$ – parametric family; $\frac{d\varphi}{d\mu}$ – Radon–Nikodym's derivative. Therefore, additional possibilities for highlighting the difference between function and formula defining this function are being formed. Fifthly, taking into account the high level of abstraction and the generality of the material studied within the course of TMaI, its representation is certainly carried out with the constant use of graphic illustrations, diagrams, etc. The positive effect of such approach lies in deeper understanding of the theoretical material and increasing the general share of graphic and symbolic components of the MC. It should be noted that theoretical conclusions and practical methodological recommendations concerning the use of sign and symbolic means (SSM), which were presented by N. A. Tarasenkova in the recognized monograph[5] (Tarasenkova, 2002), prove very topical in the study of TMaI too.

Firstly, we should focus on the following ideas:

1) it is always necessary to look for the optimal symbolic shell for the same content, and in the higher school it is reasonable to analyze and compare such shells;
2) suitable and well-considered use of SSM helps to remove or avoid the conflict between new and intuitively familiar abstract and logical notions;
3) in teaching and learning every mathematical discipline, including TMaI, it is important to introduce the widest range of SSM – both verbal and non-verbal;
4) it is a must to constantly pay attention to the fact that in case of similar symbols or even the same symbolic shell we can arrive at different content. The ability to identify such content by using mathematical context or even without it indicates the level of MC;
5) the formation of a developed sign and symbolic operating culture is one of the important tasks of mathematical education and higher education in particular;
6) serious mathematical activity is formed through sign and symbolic activity and with it. At the same time, verbal, logical, and visual thinking develop in a harmonious way;
7) methodically balanced and reasonably developed sign and symbolic components of TMaI are a guarantee of a conscious, psychologically comfortable study of this complex course by students-mathematicians.

10. Knowledge and understanding of methodology of mathematics. Concepts of methodology and methodology of mathematics could be approached in different ways. In our research the methodology (from Greek *μεθοδος* – way of research or cognition and *λογος* – science) is treated as:

1) the set of research methods which are applied in particular science;

[5] Tarasenkova, N. (2002). *Using sign and symbolic means in teaching mathematics. Monograph*. Cherkasy, Ukraine: "Vidlunnya-Plyus". (In Ukr.).

2) the doctrine concerning the methods of cognition and transformation of reality[6] (Goncharenko, 1997).

Concerning the methodology of mathematics, we use the interpretation of such notion as a doctrine, which considers specifics, subject and method of mathematics from the point of view of the theory of cognition and as the doctrine of the construction and structure of formal systems, and the related questions of consistency, categoricality, completeness of the corresponding axiomatics. Taking into account the above mentioned ideas concerning the understanding of the methodology of mathematics, we can state that the course TMaI offers significant opportunities for the formation of the corresponding element of the of mathematics students' MC.

Firstly, on the material of the TMaI course, it is possible to demonstrate in a well-reasonable way that mathematics is the science about mathematical structures (in TMaI few such structures are considered: spaces with measures, spaces of integrated function $L_p(X, A, \mu), 1 \leq p < \infty$; etc.). Secondly, in the course of TMaI the most important methods of mathematics (method of abstraction, axiomatic method, method of mathematical modeling, deductive method) are used in a very efficient way. For example, *measure* is a notion built by abstraction and idealization of such notions as length, area, volume, mass, etc. Introducing such notions as ring, semi-ring, σ-algebra, monotone class, measure, external measure, and charge is carried out with the use of the axiomatic method.

Notions, which are fundamental for the TMaI course and for mathematics as a whole, namely, *function, measure, integral* are the examples of more or less mediated models of real phenomena and processes (continuous measure on line simulates the continuous distribution of mass, for example, on a homogeneous string; discrete measure on line simulates the weightless string with strung beads; mathematical models of different changes, movements, processes are built with using the functions; integral helps to simulate the phenomena of accumulation, summation, unions).

Modern mathematics is the evidence of the triumph of deductive method. And it seems very natural as it is an indispensable consequence of using an axiomatic method.

[6] Goncharenko, S.Yu. (1997). *Ukrainian pedagogical dictionary.* Kyiv, Ukraine: Lybid. (In Ukr.).

Therefore, in TMaI, which is part of modern mathematics, and in the course of TMaI, the deductive methods are widely used. For example, after axiomatic definition of measure the whole set of its properties is established by using a deductive way, also the issues of self-consistency, categoricity and completeness of corresponding axiomatic are considered. At the same time, it should be noted that inductive method is also used in the TMaI course. But, the application of the inductive method isn’t so wide in comparison with deductive method. The inductive method is mainly used when the idea is being formed or as the way to motivate the introducing some abstract notions or constructions.

The notion of infinity is a cross-cutting theme in mathematics. Moreover, certain scientists and, in particular, the famous mathematician H. Weyl[7] **(1989)**, defined mathematics as the science of infinity. The clarification of the essence of such mathematical abstraction as infinity is an important mathematical and methodological task. Now, only two types of infinity are used in mathematics: potential and actual. TMaI belongs to the part of mathematics in which both of these types of infinity are recognized and fully used. In the course of TMaI, the mathematical and philosophical aspects of the concept of infinity are usually left out of consideration, while the use of both types of infinity is constantly emphasized, namely, the impossibility of constructing TMaI without using both aspects. Constant and repeated reference to the concepts of potential and actual infinity facilitates the formation of quite adequate perception of these concepts by students.

The concept of potential infinity is based on the hypothesis of potential actualization, which allows of the construction of not only such objects that could be constructed practically (at least, in principle), but also objects, which could be constructed potentially on the assumption that we possess all the required capabilities. Considering definitions, theorems, or problems where the convergence of sequences or series appears, we should understand that we are dealing with potential infinity.

The concept of actual infinity is based on the hypothesis of absolute actualization, in which the existence of any objects could be considered without any contradictions. Actual infinity suggests that elements of an infinite set, like

[7] Weyl, H. (1989). *Mathematical thinking: Transl. from Engl. and Germ.* Moscow, Russia: Nauka. (In Rus.)

elements of a finite set, can exist simultaneously. In this case, the existence of mathematical objects is not associated with the possibility of their construction; it is postulated by axioms and definitions. Therefore, when in TMaI we carry out operations with infinite sets, we are to understand that we are dealing with the actual infinity. Another conclusion is that the theory of infinite sets is impossible without the concept of actual infinity. It is evident from the mentioned above that TMaI has significant opportunities for the formation of modern ideas about the nature, place and role of the potential and actual infinities in mathematics.

Consideration of the issues of the mathematical and methodological potential of TMaI will be incomplete without mentioning Zermelo's axiom of choice. "I think days and nights about the Zermelo's axiom. If only somebody would know what this thing was!". F. A. Medvedev used these words, which belong to the outstanding mathematician M. Luzin, as an epigraph in the preface of his own book[8] (Medvedev, 2013). The axiom of the choice of Zermelo is one of the most important statments of modern mathematics. The ideas based on this theorem or on equivalent statements and the results founded by using of Zermelo's axiom are so numerous and, in most cases, so important that their removal from mathematics would completely change its face. This axiom or its equivalents are used in classical and functional analysis, theory of sets, topology, algebra, theory of probabilities, graph theory, mathematical logic, etc., up to elementary geometry and arithmetic. This axiom was one of the most discussed statements of the theory of sets in the twenty century. In mathematical literature, the comparison of research concerning the axiom of choice with studies of the axiom of parallels in geometry has long history. Yet, despite the great mathematical and cultural significance of the Zermelo's axiom, there are not so many reasons to pay it attention in university courses. Therefore, we must appreciate every such opportunity. In TMaI, such an opportunity exists – this is the construction of Lebesgue's immeasurable set, which is a self-valuable mathematical and cultural fact.

11. Knowledge and understanding of the main components of mathematics, relationships, and perspective directions of its development. The study of the TMaI course, on the one hand,

[8] Medvedev, F. A. (2013). *Early history of axiom of choice.* Moscow, Russia: LIBROKOM. (In Rus.).

gives a clear understanding concerning the corresponding branch of mathematics, and, on the other hand, highlights the interrelationships between very important branches of mathematics, related to the so-called major analysis. There are following branches of mathematics related to major analysis: set theory, mathematical analysis, complex analysis, functional analysis, and probability theory, the theory of random processes, mathematical statistics, and equations of mathematical physics, integral equations, variations calculus, and fractal analysis. The set theory serves for the TMaI as a foundation; many concepts and content lines of mathematical analysis are developed and generalized in the TMaI course, and for all other branches of mathematics TMaI creates tools, supplies ideas and methods, which, in turn, give impulses for their own development. With TMaI all these branches are interconnected, interdependent, mutually enriching and complementary, which, in the end, results in their unity. In the author's opinion, the non-commutative theory of measure, fractal analysis, and the theory of fuzzy sets, as viewed from the standpoint of TMaI, should be considered the perspective vectors of the development of modern mathematics.

15. Knowledge of the history of the appearance and development of mathematics, its main branches, biographies of famous mathematicians. "Mathematical education of a mathematician will be incomplete if each mathematical course does not include some information from the history of the development of ideas fundamental for this course. How did the idea of making mathematics an independent discipline emerge; how were its main notions developed; what famous scientists played an outstanding role in obtaining the main results? It is important to emphasize the fact that scientists from different countries took part in scientific progress. This information should be brief, but given systematically"[9] (Gnedenko, 2006). These words of the famous mathematician and educator are a great guide concerning the development of an appropriate element of the mathemaatics' students MC. Within the study of TMaI, there are many reasons to address historical and mathematical facts in order to develop the

[9] Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millennium*. Moscow, Russia: ComKniga. (In Rus.)

interest in the theory, to understand the motives and logic of the scientists makingn their discoveries, and in order to show, at least briefly, the historical and cultural background against which some particular events took place.

And, if we take into account the fact that the most prominent mathematicians of different times participated in the development of TMaI, as, for example, Newton, Leibniz, Cauchy, Riemann, Jordan, Borelean, Lebesgue, Kolmogorov, etc., it becomes clear that there are more than enough historical and mathematical materials, including biographical ones.

Analysis of educational programs and curricula of the TMaI course allows of defining five content lines, implementation of which should ensure the achievement of the main goals of the course. Such content lines include:

1) the set and theoretical line;
2) functional line;
3) the line of measure;
4) the line of convergence;
5) the line of the integral.

It seems reasonable to consider the historical and mathematical background of each of mentioned lines, highlighting the important information for students.

1. Set-theoretical line. It should be emphasizeв that:

a) set theory is the phenomenon whose impact on mathematics and science as a whole is incredible;

b) the creation of TMaI is the necessary stage in the development of mathematics, was prepared by all its previous development, and is impossible without the set theory;

c) antinomies of the theory of sets caused the crisis of the foundations of mathematics, still did not estrange most of leading mathematicians. D. Hilbert declared that “No one would banish us from the paradise created by Cantor”, and these words expressed the common opinion;

d) the overcoming of the crisis caused by antinomies was made possible with the use of the axiomatization of set theory;

e) Zermelo’s axiom of the choice is the most mysterious axiom among the set theory axioms, the relation to the axiom of choice is an important problem, the cornerstone of the philosophy of mathematics;

f) George Cantor’s great scientific exploit cannot be neglected and forgotten.

2. Functional line. It should be noted that:

a) function is the main notion of mathematics. “The notion of function is one of the main notions of modern mathematics. It was not developed immediately, but having emerged more than two hundred years ago in the famous debate about the string, this notion was the subject of profound changes, which took place in the course of fierce controversy. Since then, a continuous deepening and evolution of this notion has been taking place. Therefore, the general meaning of the concept of function can’t be covered by single formal definition; it is possible to learn the following only analyzing the main vectors of its development that is closely related to the development of science, in particular – mathematical physics”[10] (Luzin, 1959). The relevance of the words of the outstanding mathematician and teacher M.M. Luzin, which were formulated approximately 80 years ago, has not decreased in its importance over the decades;

b) even prominent mathematicians who lived at the same time (eg Borel, Ber, Lebesgue) treated the notion of function in different ways;

c) functions are the mathematical instrument for modelling a variety of processes in the society, animate and inanimate nature, and technologies;

d) new functions are being constantly developed, including those with the previously unknown nature, the arsenal of “function creation” is constantly developed;

e) social, economic, and ideological situation in the country and society influences the life and creative work of scientists, in particular those of mathematicians (for example, “the case of academician Luzin”).

3. Line of measure. It is necessary to expose the following ideas to the students:

a) the notion of measure has a long history dating back to ancient times (for example, “the method of depletion” of Archimedes);

[10] Luzin, N. N. (1959). *Collected works. Volume III.* Moscow, Russia: Publishing house of the Academy of Sciences. (In Rus.)

b) measures and charges are the functions of set, which is the function of the nature different than those of point;

c) perception of the measure as a function (unusual, but nevertheless – a function) took place much later than the perception of the function of the point[11] (Medvedev, 2006);

d) it is more correct and convenient to formulate and solve a lot of tasks of natural science in terms of function of set (not in the terms of function of point); at the same time the concepts of function of point and function of set are not contradicted, but complement each other (Radon-Nikodim's theorem is a striking example).

4. Line of convergence. It is necessary to inform the students that:

a) the notion of the limit is one of the most important notions of analysis, which is used in order to express the notions of continuity, derivative, integral, sum of a series and many other concepts;

b) mathematical analysis (analysis of infinitesimal) got a solid logical foundation only on the basis of the theory of limits and the rigorous theory of real numbers;

c) the theory of limits came a long and complicated path of development from the limit of the numerical sequence to the universal theory of Cartan's filter convergence;

d) there are many types of convergence of functional sequences: uniform (at least three kinds) convergence, generalized uniform convergence, quasi-uniform convergence of Arzel, almost everywhere convergence, convergence to an extent, convergence in mean, and others, which are the issues of great interest of mathematicians, because the representation of complex functions by using the series or sequences of simpler functions is always a matter of convergence;

e) the possibility to carry out a boundary transition under the sign of the Lebesgue's integral under the very uncomplicated conditions is one of its most important and most valuable properties;

f) the history of the types of convergence mentioned above is instructive and partly dramatic and deserves to careful study.

[11] Medvedev, F. A. (2006). *French school of the theory of functions and sets at the turn of XIX and XX centuries*. Moscow, Russia: ComKniga. (In Rus.).

5. Integral line. It should be constantly emphasized that:

a) the notion of the integral is one of the most important concepts in mathematics, it has a complex and long history of formation and development;

b) now, the theory of the integral is a powerful theory, which still intensively develops;

c) the most prominent mathematicians like Archimedes, Newton, Leibniz, Cauchy, Riemann, Lebesgue took part in the formation and development of this theory; their biographies are interesting and very instructive;

d) there is no consensus among historians of mathematics concerning the time of the formation of methods of integration (some of them try to connect it with the times of Eudoxus and Archimedes, others with earlier times, but there are historians who refer it to times of Newton and Leibniz[12] (Medvedev, 2013);

e) among the large number of concepts of integral, the most important (in order of its appearance) are the Newton’s integral, the Cauchy’s integral, the Riemann’s integral, the Stieltjes’s integral, and the Lebesgue’s integrals;

f) the development of the concept of integral occurred and still occurs under the influence of external and internal factors, the main requirements of which are: expansion of the set of integral functions; relaxation of conditions under which the boundary transition is possible under the sign of integral; extension of the class of functions that are restored on derivative with the help of indefinite integration.

20. Ability to perception and creative acquisition of new ideas, creativity. In the evaluation of the qualities of the specialist, the focus on the personality, capable of solving a wide range of tasks of the modern information society, as well as reliance on a competent and creative personality is becoming more topical. Creativity is treated as the ability to creative work in the broadest sense of meaning. According to the ideas of teachers and psychologists[13] (Krutetskii, 1968), creativity defines the creative and productive orientation of the individual, his life orientation. In this study, we considered the mathematical creativity, and therefore

[12] Medvedev, F. A. (2013). *Development of the concept of integral*. Moscow, Russia: «LIBROKOM». (In Rus.).

[13] Krutetskii, V. A. (1968). *Psychology of mathematical abilities of schoolchildren*. Moscow, Russia: Prosveshchenie. (In Rus.).

we did not oppose creativity and intelligence, creativity and mathematical abilities; on the contrary, we consider creativity as the highest manifestation of intellect, the realization of a person's creative abilities. And we also share the point of view that creativity, being an innate characterization of a person, allows of the development and self-development. Such views are formed under the influence of such scientists as J. Hadamard, A. N. Kolmogorov, V. A. Krutetskii, G. Poiya, H. Poincaré, V. M. Tikhomirov and others. The most important goals for all academic disciplines are the following: to develop the ability to learn, to perceive and to master the new ideas, to develop the creativity. The course of TMaI does not stand aside, especially because it has significant opportunities for the formation of the corresponding element of the MC of mathematics students.

We see these opportunities in the following:

1) the course of TMaI makes a significant contribution to the formation of elements 1 – 12 MC, creating the necessary conditions for the perception and creative acquisition of the new and for the manifestation of creativity in the mathematics;
2) in the course of TMaI a lot of new abstract concepts are introduced; many theorems, complex on logical and technical reasons, are studied, this influences the formation of the ability to learn, creative perception of new material;
3) the course of TMaI envisages solving a significant number of original, non-standard tasks, which, as is known, contributes to the development of mathematical creativity;
4) the TMaI course is closely connected with many mathematics subjects and uses the knowledge and skills acquired in them, which creates the systemacy, versatility, variability, impartiality and universalism in approaches to the solution of a wide variety of mathematical problems;
5) in the process of studying TMaI there is an acquaintance of students with the mathematical creativity of outstanding mathematicians, which allows them of keeping in touch with the best samples of mathematical creativity.

Concerning the formation of the elements of 13–14 and 16–19 of MC of mathematics students, the possibilities of the course of TMaI are less pronounced with taking into account its specific character. These possibilities are implemented in the “background” mode through the system of specially selected pro-mathematical historisms.

In conclusion, we suggest the following:

1. The proposed model of MC significantly helps in revealing mathematical and cultural possibilities of each educational mathematical discipline, in particular those of TMaI;

2. The analysis of the elements of mathematics students’ MC in the context of its formation in the course of TMaI shows that this course has a significant mathematical and cultural potential;

3. The author’s experience of teaching TMaI demonstrates that taking into account and using of the mathematical and cultural possibilities of the TMaI course has significant influence on the level of formation of MC of students.

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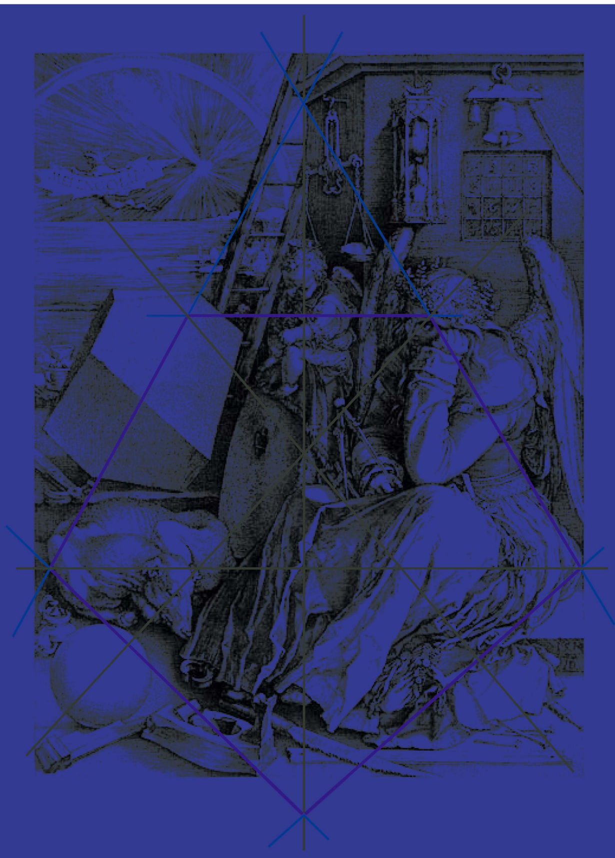

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