

One variant of the collocation-iterative method of solving integro-functional equations

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Abstract. The construction of exact solutions of integro-functional equations is possible only in some cases, so the questions of studying approximate methods for solving these equations, convergence conditions and estimating the errors are relevant. The article considers the construction of approximate solutions of linear integro-functional equations. It is shown that, under certain conditions, such solutions can be obtained by applying one variant of the collocation-iterative method to the problem. The algorithm of the method and sufficient convergence conditions are indicated.

Keywords: linear integro-functional equation, Fredholm integral equation of the second kind, collocation-iterative method, approximate solution, collocation nodes.

Introduction. Many problems of applied and theoretical nature cannot be imagined without the construction and reaserch of various mathematical models. Economics, biology, technology and many other fields of human activity are increasingly based in their research on the results obtained in the theory of operator equations. Therefore, many boundary value problems for differential equations with argument deviation are models of different natural processes. The theoretical foundations of these problems are laid in the works of A. Samoilenko, M. Perestyuk [11], A.Luchka [7] and other scientists.

Such problems are reduced to solving integro-functional equations. The exact solution of these equations can be found only in some simple cases. In more complex cases, the exact solution cannot be expressed by elementary functions. Therefore, approximate methods are important. The research of the projection-iterative method and its generalizations are devoted to the works of A. Luchka [8], N. Kurpel [4] and other. The methods of projection-iterative type include the collocation-iterative method, which arose as a result of the synthesis of the method of successive approximations and the method of collocation. It is quite effective in terms of finding approximate solutions of different types of integro-functional equations [1, 2]. The application of the collocation-iterative method for solving integro-functional equations is the main object of study of this article.

Analysis of recent research and publications. The research and theoretical substantiation of the collocation-iterative method with respect to one-dimensional and multidimensional Fredholm integral equations, the study of the rate of convergence of the method depending on the smoothness of the initial data are devoted to the work of A.Y. Luchka [7, 9] and E.M. Lutsev [6].

The application of the collocation-iterative method to find the solution of the boundary value problem for ordinary differential equations with parameters is devoted to the works of Poselyuzhna V.B. and Semchyshyna L.M. [10], Kucheruk T. [5]. In these works, the question of applying the collocation-iterative method (both stationary and nonstationary) to find the solution of the boundary value problem for ordinary differential equations with parameters, for differential equations with small nonlinearity, and for the boundary value problem for ordinary differential equations with momentum influence and

parameters. The application of a modified collocation-iterative method for solving integral Fredholm equations of the 2nd kind and differential equations with small non-linearity is also substantiated.

Despite the considerable many of work devoted to various aspects of the theory of approximate methods for solving different types of operator equations, the task of obtaining compatibility conditions and constructing new methods for finding approximate solutions of such equations is still far from being fully solved.

The purpose and objectives of the research. Construction and research of the studied publishing developments of integra-functional levels by collocation-interactive method. Describe the conditions and rate of convergence of this method and approximation errors.

Materials and methods

In space $L_2(a;b)$ – real and measuring on the interval $(a;b)$ of functions, summable with a square, consider the integro-functional equation form

$$y(x) - p(x)y(h(x)) = f(x) + \int_a^b K(x;t)y(t)dt, x \in (a;b), \quad (1)$$

$$y(x) = 0, x \notin (a;b),$$

where $f(x)$ – known, and $y(x)$ – unknown functions from $L_2(a;b)$. Regarding the functions $h(x), p(x), K(x;t)$ we assume that they, according to on the interval $[a;b]$ and on the squared $[a;b]^2$ satisfy the conditions:

$$|p(x)| \leq \bar{p} < \infty, \quad (2)$$

$$h(x) - \text{differentiable on } [a;b] \text{ and}$$

$$h'(x) \geq l > 0, x - h(x) \geq \sigma > 0, \quad (3)$$

$$\iint_{aa}^{bb} K^2(x;t) dx dt = B^2 < \infty. \quad (4)$$

We show that equation (1) under conditions (2) – (4) can be reduced to the integral Fredholm equation of the second kind. Next to the integral completely continuous operator K which has a form

$$(Kv)(x) = \int_a^b K(x;t)v(t)dt, \quad \forall v(x) \in L_2(a;b),$$

we consider the operator S such as

$$(Sv)(x) = \begin{cases} v(x), x \in [a; h^{-1}(a)], \\ v(x) - p(x)v(h(x)), x \in [h^{-1}(a); b], \end{cases} \quad (5)$$

where $v(x)$ – arbitrary function from $L_2(a;b)$.

Note that this operator, like the operator K , acts from $L_2(a;b)$ to $L_2(a;b)$. It is easy to show that the operator S linear. Conditions (2), (3) guarantee its limitation. Really,

$$\|S\| = \sup \frac{(Sv)(x)}{v(x)} \leq 1 + \left| \frac{p^2(x)}{h'(x)} \right|^{\frac{1}{2}} \leq 1 + \frac{\bar{p}}{\sqrt{l}} < \infty,$$

where \sup taken on $v(x) \in L_2(a;b), v(x) \neq 0$.

These conditions suggest that the operator S reversible. Inverse operator looks

$$(S^{-1}v)(x) = \begin{cases} v(x), x \in [a; h^{-1}(a)], \\ v(x) + \sum_{i=1}^s v(h^i(x)) \prod_{k=0}^{i-1} p(h^k(x)), \end{cases} \quad (6)$$

$$x \in \Delta_s, s = \overline{1, m}.$$

Here, as in the future,

$$\Delta_s = [c_{s-1}; c_s],$$

$$c_0 = a, c_s = h^{-1}(c_{s-1}), c_m = b,$$

$$h^k(x) = h(h^{k-1}(x)), s = \overline{1, m}.$$

So, in other words, expression (6) – is a solution of a functional equation

$$y(x) - p(x)y(h(x)) = u(x), x \in [a; b],$$

$$y(x) = 0, x \notin [a; b],$$

(where $u(x)$ – known, $y(x)$ – unknown functions)

using the method of steps. Condition (3) guarantees the fact that the number of steps m are finite and

$$m \leq \left\lceil \frac{b-a}{\sigma} \right\rceil.$$

It is easy to make sure that the operator S^{-1} , as well as the operator S , linear and limited. Thus, given the above considerations, we can consider equation (1) as an operator equation

$$(Sy)(x) = f(x) + (Ky)(x), \quad (7)$$

where $f(x)$ – known, $y(x)$ – unknown functions

from $L_2(a;b)$.

Let $(Sy)(x) = u(x)$, then $y(x) = (S^{-1}u)(x)$ and we are of the equation (7) go to the equation

$$u(x) = f(x) + (Tu)(x). \quad (8)$$

Operator $T = KS^{-1}$ is Fredholm as a superposition of Fredholm and linear limited operators. In other words, using the above-mentioned substitution, we transform the integro-functional equation (1) into the integral Fredholm equation of the second kind

$$u(x) = f(x) + \int_a^b T(x;t)u(t)dt, \quad (9)$$

with a completely continuous integral operator T and

$$T(x;t) = \begin{cases} K(x;t) + \sum_{i=1}^{m-s} K(x; (h^{-1})^k(t)), t \in \Delta_s, \\ K(x;t), t \in (c_{m-1}; b), s = \overline{1, m-1}, x \in (a; b), \end{cases} \quad (10)$$

where $(h^{-1})^k(t) = h^{-1}((h^{-1})^{k-1}(t))$.

We can apply the method of successive approximations, collocation and some collocation-iterative methods to the problem (1) [3, 6]. Moreover, the most effective of convergence conditions is the last method, accordingly, the previous two methods can be considered as partial cases [5].

Let $\{\varphi_i(x)\}$ – linearly-independent system of functions from $L_2(a;b)$. Suppose, that based on the initial approximation $y_0 \in L_2(a;b)$ and functions $s_0(x)$ such as

$$y_0(x) - p(x)y_0(h(x)) = s_0(x), x \in [a; b],$$

$$y_0(x) = 0, x \notin [a; b],$$

we found an approximation $y_{k-1}(x)$ and function $s_{k-1}(x)$. The next approximation $y_k(x)$ we find from the functional equation

$$y_k(x) - p(x)y_k(h(x)) = f(x) + \int_a^b K(x;t)z_k(t)dt, x \in [a; b], \quad (11)$$

$$y_k(x) = 0, x \notin [a; b],$$

which

$$z_k(x) = y_{k-1}(x) + w_k(x), \quad w_k(x) = \sum_{j=1}^n a_j^k \eta_j(x). \quad (12)$$

Unknown coefficients $a_j^k = a_j^k(n)$ determined from the condition

$$r_k(x_i) = 0, i = \overline{1, n}, \quad (13)$$

where

$$r_k(x_i) = f(x) + \int_a^b K(x;t)z_k(t)dt - z_k(x) + p(x)z_k(h(x)), \quad (14)$$

$$x_i \in [a; b] - \text{collocation nodes}, i = \overline{1, n}.$$

System of functions $\{\eta_j(x)\}, j = \overline{1, n}$ are determined from functional equations

$$\eta_j(x) - p(x)\eta_j(h(x)) = \varphi_j(x), x \in [a; b], \quad (15)$$

$$\eta_j(x) = 0, x \notin [a; b].$$

Results and discussion. The algorithm is given by replacement $u_k(x) = (Sy_k)(x)$, $\alpha_k(x) = (Sw_k)(x)$ reduces to a collocation-iterative method of solving integral equa-

tion (9) with a kernel having the form (10). Therefore, the following statement will take place.

Theorem. If one is not a point of the spectrum of the integral operator T equation (9), and system of functions $\{\varphi_i(x)\}$ is complete in space $L_2(a; b)$, then there will be such a number n , that method (11) – (15) will be convergent, and convergence rate will increase with increasing n .

Here are some sufficient conditions for the convergence of the method (11) – (15).

Suppose, as mentioned above, functions $p(x), h(x)$ and $K(x; t)$ satisfy the conditions (2) – (4). System of functions $\{\varphi_i(x)\}$ is complete and orthogonal in $L_2(a; b)$. The answer to the question of which n the above collocation-iterative method will be convergent, can be obtained using the algorithm. The essence of these algorithm is follows.

First, we construct functions by a recurrent formula $U'_s(x; t) = U_{s-1}(x; t) - \alpha_s E_s(x) \omega_s(t), s = 1, 2, \dots,$ (16) where

$$E_s(x) = \int_a^b U_{s-1}(x; t) \eta_s(t) dt, \quad \alpha_s^{-1} = \int_a^b \varphi_s^2(t) dt, \quad (17)$$

$$\omega_s(t) = \lambda_s(t) - \int_a^b U_{s-1}(x; t) \varphi_s(x) dx, \quad (18)$$

$$\lambda_s(t) = \varphi_s(t) - p(h^{-1}(t)) \varphi_s(h^{-1}(t)), t \in [a; b], \quad (19)$$

$$\varphi_s(t) = 0, \text{ when } t \notin [a; b].$$

We take $U_0(x; t) = K(x; t)$, and we find functions $\eta_s(x)$ by solving the functional equation (15).

After that we write function $U'_s(x; t) = U'_s(x; t) - \varphi_s(x) \beta_s(t), s = 1, 2, \dots,$ (20)

where

$$\beta_s(t) = \alpha_s \int_a^b U'_s(x; t) \varphi_s(x) dx. \quad (21)$$

Next, we solve the functional equation $L_s(x; t) - p(h^{-1}(t)) L_s(x; h^{-1}(t)) = U'_s(x; t), t \in [a; b],$ (22)

$$L_s(x; t) = 0, t \notin [a; b],$$

and we find the function $L_s(x; t), s = 1, 2, \dots$

Then calculate

$$q_s^2 = \int_a^b \int_a^b L_s^2(x; t) dx dt. \quad (23)$$

It turns out that if $q_s < 1$, then we come to the conclusion that method (11) – (15) will be convergent. If $q_s \geq 1$, then the question of the convergence of the method remains open. Gradually increasing the number of coordinate functions $\varphi_i(x), i = s + 1, s + 2, \dots$, according to the same recurrent formulas (16) – (23) we continued the calculation until we come to such n , that inequality $q_s < 1$ will be.

The convergence rate of collocation-iterative method will be characterized by inequality

$$\|y^*(x) - y_k(x)\| \leq C_n \cdot q_n^{k-1} \cdot \|y^*(x) - y_0(x)\|, \quad (24)$$

C_n – some positive constant, and $y^*(x)$ – exact solution of the integro-functional equation

$$y(x) - p(x) y(h(x)) = f(x) + \int_a^b K(x; t) y(t) dt, x \in (a; b), \quad (25)$$

$$y(x) = 0, x \notin (a; b),$$

where $f(x)$ – known, a $y(x)$ – unknown functions from $L_2(a; b)$.

This estimate can be used to finding the number of iterations, which necessary to reduce the initial error by the required number of times.

There also will be a constructive estimate:

$$\|y^*(x) - y_k(x)\| \leq \frac{p_n q_n^{k-s}}{1 - q_n} \cdot \|y_s(x) - z_s(x)\|, \quad (26)$$

where p_n – some positive number, $1 \leq s \leq k$, and $z_s(x)$ looks like (12).

Direct calculations according to method (11) – (15) should be performed as follows. First, we define a linearly-independent system of functions $\{\varphi_j(x)\}, j = \overline{1, n}$ and by the formula (15) find the system $\{\eta_j(x)\}$. After that we find functions

$$K_j(x) = \int_a^b K(x; t) \eta_j(t) dt, j = \overline{1, n}. \quad (27)$$

Next by the formula

$$\beta_{ij} = \varphi_j(x_i) - K_j(x_i), \quad (28)$$

we can find the elements of the matrix Λ (тыт $x_i, i = \overline{1, n}$ – collocation nodes). Then we find the inverse matrix Λ^{-1} and move on to main calculations.

So, let the approximation $y_{k-1}(x)$ and according to function $s_{k-1}(x)$ already set. Then we perform the iteration

$$v_k(x) = f(x) - \int_a^b K(x; t) y_{k-1}(t) dt, \quad (29)$$

and we find a discrepancy

$$\varepsilon_k(x) = v_k(x) - s_{k-1}(x). \quad (30)$$

According to the formula

$$b_i^k = \varepsilon_k(x), \quad (31)$$

we find the coordinates of the vector b_i^k .

We write the equation $\Lambda \bar{a}_k = \bar{b}_k$ and we find its solution $\bar{a}_k = \Lambda^{-1} \bar{b}_k = (a_1^k, a_2^k, \dots, a_n^k).$ (32)

We construct functions

$$s_k(x) = v_k(x) + \sum_{j=1}^n a_j^k K_j(x), j = \overline{1, n}. \quad (33)$$

Then the approximate solution found from the functional equation

$$y_k(x) - p(x)y_k(h(x)) = s_k(x), x \in (a; b),$$

$$y_k(x) = 0, x \notin (a; b). \quad (34)$$

Conclusions. The results obtained in the article expand the scope of the collocation-iterative method and enrich the theory of integro-functional equations. The developed algorithm of the method can be used to find solutions to specific mathematical problems that come from physics, biology, economics and other scientific fields. It is shown that the collocation-iterative method of solving integro-functional equations makes it possible to find solutions to

the problem. When using this method, at each step of the iteration there is a need to solve a linear system of algebraic equations. Since the basic matrix of the system is nondegenerate and the same for each iteration step, it is advisable to find the inverse matrix at the beginning of this process and then use it step by step to find approximate solutions. It should be kept in mind that the obtained results of the convergence method and constructive estimates of the errors of the method are important.

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