Lebesgue structure and properties of the inversor of digits of Q_s-representation for fractional part of real number

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Abstract. We construct a $s-1$ parametric family of continuous purely singular functions (if $q_i \neq q_{s-1-i}$, $q_i > 0$, $\sum_{i=0}^{s-1} q_i = 1$) with self-similar properties using Q_s-representation of real number that is generalization of classic s-adic representation. We generalize well-known results and study structural, fractal, self-affine and integral properties.

Keywords: *Q_s-representation of real numbers, inversor of Q_s-representation digits of real number, self-affine set, monotone function, singular function.*

Introduction. Many continuous on segment $[0;1]$ functions have fractal properties. For some, it is fractality of the graph (the graph is a fractal curve of space P_2) [5, 10], for others it is fractality of the level sets [5, 6], and property of functions to keep dimension of all Borel sets [3, 11], etc. Among functions with complex local structure and fractal properties, at present time continuous monotonic functions [6, 11, 12] and non-monotonic functions [2, 7, 8] are of particular interest. Its theoretical research is carried out in different directions (topological-metric and fractal analysis of levels, studying of sets of different features, structural analysis of graphs, having of properties of self-similarity, etc.). Along with this, the functions appear more often in various models of real objects, processes and phenomena [1].

Rapid development of mathematics led to the necessity of using different number systems and different representations of real numbers, in particular, non-traditional ones. This systems allow us to describe classes of fractal sets, functions, probability distributions, and investigate objects with a complex local structure. In this paper we use one of the following encodings: Q_s-representation of real numbers [6, 10].

Object of study. In the paper we study function that depends on the parameter $q_0, q_1, \dots, q_{s-2}$, and use for its assignment is so-called Q_s-representation $x \in [0,1]$ [6]. This representation is encoding of number with finite alphabet $A \equiv \{0,1,\dots,s-1\}$ and is generalization of the classical s-adic representation of real numbers.

Let $Q_s = \{q_0, q_1, \dots, q_{s-1}\}$ be ordered set of positive real numbers such that $\sum_{i=0}^{s-1} q_i = 1$; $\beta_0 = 0$, $\beta_k = \sum_{i=0}^{k-1} q_i$.

Theorem 1. [6] *For an arbitrary $x \in [0;1]$ there exists a sequence (α_n), $\alpha_n \in A_s$ such that*

$$x = \beta_{\alpha_1} + \sum_{k=2}^{\infty}\left[\beta_{\alpha_k}\prod_{j=1}^{k-1} q_{\alpha_j}\right] = \Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_n\dots}. \qquad (1)$$

The series (1) is called Q_s-image of the number x, and the abbreviated (symbolic) record $x = \Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_n\dots}$ is called its Q_s-representation.

The period in Q_s-representation of the number (if it exists) is denoted by the parentheses. There are numbers that have two Q_s-representation. These are numbers with period (0) or $(s-1)$, moreover

$$\Delta^{Q_s}_{c_1\dots c_{m-1}c_m(0)} = \Delta^{Q_s}_{c_1\dots c_{m-1}[c_m-1](s-1)}.$$

These numbers are called Q_s-rational, set of Q_s-rational numbers is countable. The rest of the numbers are called Q_s-irrational.

Definition 1. *Function defined $[0;1]$ by equality*

$$I(x) = I(\Delta^{Q_s}_{\alpha_1(x)\alpha_2(x)\dots\alpha_k(x)\dots}) = \Delta^{Q_s}_{[s-1-\alpha_1(x)][s-1-\alpha_2(x)]\dots[s-1-\alpha_k(x)]\dots}, \qquad (2)$$

is called an inversor I of digits the Q_s-representation of a real number (or simply inversor).

Function denoted by equality (2) is a generalization of the function I that was investigated in [11, 9] where $s = 3$, and, as it turned out, is one of the brilliant representatives of the functions retaining the digit 1 in Q_3-representation of the argument [12].

Research results. Since Q_s-rational numbers have two distinctly different representation, there is a need to justify the correctness of definition of inversor. Let for different Q_s-rational meanings

$$x_1 \equiv \Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_{k-1}\alpha_k(0)} = \Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_{k-1}[\alpha_k-1](s-1)} \equiv x_2$$

function, respectively, acquires the meanings:

$$I(x_1) = I(\Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_{k-1}\alpha_k(0)}) = \Delta^{Q_s}_{[s-1-\alpha_1][s-1-\alpha_2]\dots[s-1-\alpha_{k-1}][s-1-\alpha_k](s-1)} = \frac{\beta_{s-1}}{1-q_{s-1}}\prod_{j=1}^{k} q_{[s-1-\alpha_j]} = \prod_{j=1}^{k} q_{[s-1-\alpha_j]},$$

$$I(x_2) = I(\Delta^{Q_s}_{\alpha_1\alpha_2\dots\alpha_{k-1}[\alpha_k-1](s-1)}) = \Delta^{Q_s}_{[s-1-\alpha_1][s-1-\alpha_2]\dots[s-1-\alpha_{k-1}][s-1-\alpha_k](0)} = \prod_{j=1}^{k} q_{[s-1-\alpha_j]}.$$

Evidently $I(x_1) = I(x_2)$, the function is correctly defined in each Q_s-rational points, and hence at each point of the segment $[0;1]$.

From these considerations, the following statement is evident.

Lemma 1. *For the growth of the* $\mu_I(\Delta^{Q_s}_{c_1c_2\ldots c_n}) \equiv I(\Delta^{Q_s}_{c_1c_2\ldots c_n(s-1)}) - I(\Delta^{Q_s}_{c_1c_2\ldots c_n(0)})$ *of the function* I *in the cylinder* $\Delta^{Q_s}_{c_1c_2\ldots c_n} = [\Delta^{Q_s}_{c_1c_2\ldots c_n(0)}; \Delta^{Q_s}_{c_1c_2\ldots c_n(s-1)}]$ *equality*

$$\mu_I(\Delta^{Q_s}_{c_1c_2\ldots c_n}) = -\prod_{j=1}^{n} q_{[s-1-c_j]}.$$

Theorem 2. *Inversor* I *of digits of* Q_s *-representation of real number* $[0,1]$ *is a function that:*

1) have continuous line at each point of segment $[0;1]$ *and acquires all values from this range;*

2) is strictly monotonic decreasing on a segment $[0;1]$.

Proving. 1) To prove continuity it is enough to show that for an arbitrary point $x_0 \in [0;1]$ holds $\lim\limits_{x\to x_0} |f(x) - f(x_0)| = 0$.

The proof will be carried out separately for cases where x_0 is a Q_s -rational and Q_s -irrational number.

Let x_0 be some Q_s -irrational number. Then, for an arbitrary number $x \in [0;1]$ such that $x \to x_0$ it is possible to specify cylinder of rank $n = n(x)$:

$$\begin{cases} \alpha_j(x) = \alpha_j(x_0) \text{ where } j < n, \\ \alpha_n(x) \neq \alpha_n(x_0), \end{cases}$$

and the condition $x \to x_0$ is equivalent to the condition $n \to \infty$. Then

$$I(x_0) = \Delta^{Q_s}_{[s-1-\alpha_1(x_0)][s-1-\alpha_2(x_0)]\ldots[s-1-\alpha_{n-1}(x_0)][s-1-\alpha_n(x_0)][s-1-\alpha_{n+1}(x_0)]\ldots[s-1-\alpha_{n+k}(x_0)]\ldots},$$

$$I(x) = \Delta^{Q_s}_{[s-1-\alpha_1(x)][s-1-\alpha_2(x)]\ldots[s-1-\alpha_{n-1}(x)][s-1-\alpha_n(x)][s-1-\alpha_{n+1}(x)]\ldots[s-1-\alpha_{n+k}(x)]\ldots}.$$

So

$$I(x) - I(x_0) = \left| \sum_{i=n}^{\infty} \Big(\beta_{[s-1-\alpha_i(x)]} \prod_{j=1}^{i-1} q_{[s-1-\alpha_j(x)]}\Big) - \sum_{i=n}^{\infty} \Big(\beta_{[s-1-\alpha_i(x_0)]} \prod_{j=1}^{i-1} q_{[s-1-\alpha_j(x_0)]}\Big) \right| \le$$

$$\le \prod_{j=1}^{n-1} q_{[s-1-\alpha_j(x_0)]} (\beta_{s-1} + \beta_{s-1} q_{s-1} + \ldots) = \prod_{j=1}^{n-1} q_{[s-1-\alpha_j(x_0)]} \le \max\{q_0, q_1, \ldots, q_{s-1}\} \to 0 \quad \text{where} \quad n \to \infty,$$

which proves the continuity of the function I in the Q_s -irrational points.

The case when x_0 is Q_s -rational number is reduced to the previous one. To prove continuity of function I on the left, it is necessary to use Q_s -representation of the point x_0 with period $(s-1)$, and to the right is to use Q_s -representation of the point with period (0).

2) We prove that the function is monotonic decreasing on $[0;1]$.

Let us consider two numbers $x_1 = \Delta^{Q_s}_{\alpha_1^{(1)}\alpha_2^{(1)}\ldots\alpha_n^{(1)}\ldots}$, $x_2 = \Delta^{Q_s}_{\alpha_1^{(2)}\alpha_2^{(2)}\ldots\alpha_n^{(2)}\ldots}$, $x_1 < x_2$. Then exists a positive integer k such that: $\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_1, \alpha_2^{(1)} = \alpha_2^{(2)} = \alpha_2, \ldots, \alpha_{k-1}^{(1)} = \alpha_{k-1}^{(2)} = \alpha_{k-1}$, but $\alpha_k^{(1)} < \alpha_k^{(2)}$. Then $s-1-\alpha_k^{(1)} > s-1-\alpha_k^{(2)}$ and

$$I(x_1) - I(x_2) = \Delta^{Q_s}_{[s-1-\alpha_1^{(1)}][s-1-\alpha_2^{(1)}]\ldots[s-1-\alpha_n^{(1)}]\ldots} - \Delta^{Q_s}_{[s-1-\alpha_1^{(2)}][s-1-\alpha_2^{(2)}]\ldots[s-1-\alpha_n^{(2)}]\ldots} =$$

$$= \left(\beta_{[s-1-\alpha_k^{(1)}]} + \beta_{[s-1-\alpha_{k+1}^{(1)}]} q_{[s-1-\alpha_k^{(1)}]} + \ldots - \beta_{[s-1-\alpha_k^{(2)}]} - \beta_{[s-1-\alpha_{k+1}^{(2)}]} q_{[s-1-\alpha_k^{(2)}]} - \ldots \right) \prod_{j=1}^{k-1} q_{[s-1-\alpha_j]} > 0.$$

Consequently, if $x_1 < x_2$, then $I(x_1) > I(x_2)$, that is, the inversor of digits of Q_s -representation of a real number is a monotonic decreasing function on the segment $[0;1]$. □

Definition 2. *Different from a constant continuous function of bounded variation, the derivative of which is almost everywhere (in the sense of Lebesgue measure) is zero, is called singular.*

For any continuous function of bounded variation $f(x)$, Lebesgue's theorem [4, p. 248] states the existence and uniqueness of representation

$$f(x) = \varphi(x) + r(x), \qquad (3)$$

where $\varphi(x)$ is a absolutely continuous function and $\varphi(a) = f(a)$, and $r(x)$ is a singular function.

Equality (3) is called *the Lebesgue structure of the continuous function* $f(x)$ *of a bounded variation*.

The Lebesgue structure of the function I is solved in the following theorem.

Theorem 3. *If* $q_i \neq q_{s-1-i}$, *where* $i \in A_s$, *then the inversor* I *is a purely singular function.*

Proving. Since function I is continuous and decreasing, according to the well-known Lebesgue theorem, it has almost everywhere (in the sense of Lebesgue measure λ) finite derivative. Let A be the set of points $x \in [0,1]$ with exist $I'(x)$, B be a set of normal numbers [6] in Q_s -representation. Since Lebesgue measure $\lambda(A) = \lambda(B) = 1$, then $\lambda(A \cap B) = 1$. Let's show that $I'(x) = 0$ for any $x = \Delta^{Q_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \in A \cap B$. Because $x \in B$, then $\nu_0(x) = q_0$, $\nu_1(x) = q_1, \ldots, \nu_{s-1}(x) = q_{s-1}$.

Because

$$I'(x) = \lim_{n\to\infty} \frac{\mu_I\left(\Delta^{Q_s}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)}\right)}{|\Delta^{Q_s}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)}|} = \lim_{n\to\infty} \frac{\prod_{i=1}^{n} q_{[s-1-\alpha_i]}}{\prod_{i=1}^{n} q_{\alpha_i}} = \lim_{n\to\infty} \frac{q_0^{N_{s-1}(x,n)} \cdot \ldots \cdot q_{s-1}^{N_0(x,n)}}{q_0^{N_0(x,n)} \cdot \ldots \cdot q_{s-1}^{N_{s-1}(x,n)}} =$$

$$= \lim_{n\to\infty}\prod_{i=0}^{s-1}\left(\frac{q_i}{q_{s-1-i}}\right)^{N_{s-1-i}(x,n)-N_i(x,n)} = \lim_{n\to\infty}\prod_{i=0}^{s-1}\left(\left(\frac{q_i}{q_{s-1-i}}\right)^{\frac{N_{s-1-i}(x,n)-N_i(x,n)}{n}}\right)^n =$$

$$= \lim_{n\to\infty}\prod_{i=0}^{s-1}\left(\left(\frac{q_i}{q_{s-1-i}}\right)^{\nu_{s-1-i}(x)-\nu_i(x)}\right)^n = \lim_{n\to\infty}\prod_{i=0}^{s-1}\left(\left(\frac{q_i}{q_{s-1-i}}\right)^{q_{s-1-i}-q_i}\right)^n.$$

Obviously, for every $i \in A_s$ it is true $\left(\frac{q_i}{q_{s-1-i}}\right)^{q_{s-1-i}-q_i} < 1$, therefore

$$I'(\Delta^{Q_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \lim_{n\to\infty}\prod_{i=0}^{s-1}\left(\left(\frac{q_i}{q_{s-1-i}}\right)^{q_{s-1-i}-q_i}\right)^n = 0$$

and the function I is a singular function. □

We also investigated the fractal and integral properties of the inverter I which are solved in the following theorems.

Theorem 4. *Inversor I keeps Hausdorff-Besicovitch dimension, that is, the set and its image have the same dimension if and only if $q_i = q_{s-1-i}$ for all $i \in A_s$.*

Theorem 5. *The graph $\Gamma_I = \{(x, I(x)) : x \in [0,1]\}$ of a function I is a self-affine set, namely*

$$\Gamma_I = \bigcup_{i=0}^{s-1}\phi_i(\Gamma_I) \equiv \phi(\Gamma_I),$$

where ϕ_i is affine transformation.

Theorem 6. *If $s = 7$ for for Riemann integral for inversor I, holds equation:*

$$\int_0^1 I(x)dx = 1 - \frac{q_0^2 + q_1^2 + q_2^2 + 4(q_0q_1 + q_0q_2 + q_1q_2) + 2(q_0q_3 + q_1q_3 + q_2q_3)}{1 - 2q_0^2 - 2q_1^2 - 2q_2^2 - q_3^2}.$$

Conclusion. In this paper we investigate the Lebesgue structure of the digital inversor, the Q_s-representation of numbers, we prove that it is a bijective reflection $[0;1] \to [0;1]$, and that function graph is a self-affine set. We also find conditions when Hausdorff-Besicovitch dimension keeps and calculate the integral.

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Лебеговска структура и свойства инверсора цифр Q_s-изображения действительных чисел

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Аннотация. С помощью Q_s-изображения действительных чисел, которое является обобщением классического s-го изображения, конструируется $s-1$-параметрическая семья непрерывных чисто сингулярных функций (при условии $q_i \neq q_{s-1-i}$, $q_i > 0$, $\sum_{i=0}^{s-1} q_i = 1$) с автомодельными свойствами. Обобщаются известные результаты и изучаются структурные, фрактальные, самоафинные и интегральные свойства.

Ключевые слова: *Q_s-изображение действительного числа, инверсор цыфр Q_s-изображения действительных чисел, самоаффинное множество, сингулярная функция, монотонная функция.*