The Use of Paradoxes of Probability Theory in Teaching Students of Physical Specialities

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Abstract. The paper proposed recommendations for the use of paradoxes in probability theory and mathematical statistics for students of physical faculties of pedagogical universities; the range of probability of paradoxes and their physical interpretations.

Keywords: *paradox, counterexample, teaching methodology, theory of probability and mathematical statistics, students of physical specialties*

Introduction. Proof of mathematical statements and their denials – a daily phenomenon in the life of every student-physicist. It is mathematical statements are the apparatus by means of which we prove a number of new physical laws and phenomena. Mathematics is widely used in proving physical hypotheses. So important is the ability of the teacher of mathematics to show students-physicists applied focus of their subject. Easy enough to make it to classes on probability theory and mathematical statistics because the knowledge, abilities and skills obtained during the study of this discipline, contribute to a better understanding of physics in general and theoretical physics in particular. After all, the latter discipline allows the student to become a teacher and researcher now and in the future – scientists, and just look at the physics and mathematics as a whole.

The use of counterexamples, paradoxes and sophisms dealt with in the works of G. Sekeia [6], Y. Goncharenko [2], B. Gelbaum [1], O. Martynenko [4], O. Kuzhelia [3].

A lot of new statements in probability theory are often intuitive, especially when teachers (and students) lead a large number of examples which confirm this statement. However, in order to refute some of the allegations, do not use examples and counterexamples. "Counterexamples referred to as examples which disprove certain allegations. The difference between examples and counterexamples is that the examples confirm the general position and counterexamples illustrate the fallacy and considered a classic by means of the negation of the hypothesis" [4]. However, students were able to independently lead the counterexamples, it is necessary to develop critical thinking and create problem situation in the classroom.

Equally important in the implementation of problem methods of training is the role of paradoxes. Their use contributes to the motivation and activization of informative activity, a better understanding of theoretical material, forms the ability of critical thinking, analysis, synthesis and evaluation of certain theoretical principles and their applications. Unlike counterexamples of the use of the paradox requires from the students high level of proficiency in a theoretical material, a clear knowledge and understanding of basic concepts and facts, understand the nuances of their application.

Since elementary school teachers use paradoxical problems in logic class, and later in secondary school it is paradoxes and sophisms help engage your students, increase their interest in learning mathematics, nurture, care and teach them to think critically. Also considerable is the role of paradoxes in University education during the study of mathematics. Especially future teachers. It should be noted that for future teachers interesting enough (and such that contribute to the realization of intersubject connections) there are mathematical paradoxes that can be interpreted in terms of other Sciences (academic disciplines).

The *purpose* of this article is to present some of the paradoxes that can be used during lessons on probability theory and mathematical statistics for students of physical faculties of pedagogical universities.

Presentation of the basic material. In science, the paradoxes are incompatible (opposite) approval to justify each of which there are plausible arguments, often serve as specific stimuli to the search for truth, the development of new theories and methods of knowledge

Since the term "paradox" is widely used in modern science and in everyday life, we first recall how it is defined. In particular, in the Academic explanatory dictionary of the Ukrainian language [7] a paradox is:

1) opinion, judgment, sharply diverging with the usual, customary, and contrary to (sometimes only seemingly) sober mind;

2) a surprising phenomenon that does not meet conventional scientific views.

These are some examples of paradoxes, which can be used during lessons on probability theory and mathematical statistics for students of physical specialties with the aim of creating problem situations, thereby forming scientific world outlook and encouraging scientific research.

The paradox of the dice. "Gambling" in a world of physical particles [6]

Right roll the dice tossing with equal chances of falling on any of the faces 1, 2, 3, 4, 5 or 6. In the case of tossing of two dices sum of numbers lies in the interval from 2 to 12. 9 and 10 can be obtained in two different ways: $9 = 3 + 6 = 4 + 5$ and 10=4+6=5+5. In the task with three cubes and 9, and 10 can be obtained in six ways. Why, then 9 appears most often when rolling two dice, and 10, when you throw three?

The explanation for the paradox

When solving this problem it is necessary to fix the order of the numbers, because otherwise all the results will be equally-possible and the use of the classical approach to determining probabilities will be incorrect. In the case of two bones 9 and 10 can come out as follows: 9=3+6=6+3=4+5=5+4 and 10=4+6=6+4=5+5. This means that in the problem with two blocks 9 can be obtained in four ways, while 10 is only three. Accordingly, the chances to get 9 large. (Since the two dice gives 6×6=36 different pairs of numbers equally-possible, the chances are equal 4/36 9, and 10 – 3/36.) In the case of three dice, the situation is reversed: 9 you can get 25 ways, 10 – 26 different ways. So 10 is more likely than 10.

For students of physical specialties can offer interesting *physical interpretation* of this paradox.

The dice associated with research physicists of the 19th and 20th centuries. If we assume that instead of a die are dealing with physical particles, each face of the cube will correspond to the phase the cell in which the particle is random and which characterizes the state of the particle. Then the dice *model* is equivalent to the *Maxwell-Boltsmann* distribution for a physical particle. In this model, which is used for gas molecules, each particle with an equal chance to hit any cell, so when many equally-possible results should be taken into account, and in the coast. There is another *model of a Bose-Einstein* condensate, in which particles are inseparable, so when counting equally-possible results order to take into account is not necessary. That is the essence of our paradox lies in the fact that the dice game is described by the Maxwell-Boltzmann rather than Bose-Einstein. You should pay attention to students that none of these models is not valid for bound electrons, as in this case, each cell can be no more than one particle. Using the terminology of the dice, we can say that if one dice rolled a 6, then another 6 to fall out. And this is a *model of Fermi-Dirac*. The question arises about the application of the model of the three specified (and in addition to these three there are many more) is correct in a particular situation. Generally speaking, we cannot choose a specific model, based only on logical considerations. In most cases, to resolve this issue allow the experience or observation. However, in the case of dice it is obvious that the correct model is Maxwell-Boltzmann, and in this situation this is what we need [6].

This paradox it is useful to consider when studying the classical approach to determining the probability of a random event, after the introduction of such concepts as: the space of elementary events, a σ-algebra of random events, probability, probability space, equally-possible events, classical approach to determining probabilities.

The paradox of mortality Galilee. Anti-aging world of atoms and words [6].

It is known that Galileo investigated the lifespan, in particular made up the so-called life table, according to which the average duration of life is 26 years and with equal chances to live more than 8 years and die at the age of 8 years, i.e. the events “live less than 8 years” and “live more than 8 years” are equally-possible. The results of Galileo lead to paradoxical consequences.

Suppose that we investigate the lifetime of two people. Suppose that in the first case, one person has lived to 2 years and the other to 50 years. In this case, the average life expectancy and the probability distribution of mortality consistent with the findings of Galileo. Now, suppose that two people, one lived to be 2 years and the other to 98 years. In this case the probability distribution has not changed, and the average life expectancy increased to 50 years. That is, if in a large sample of only one person reaches the age of Methuselah (in the old Testament one of the forefathers of mankind, famous for their longevity and lived to be 969 years), average age significantly increased, but their life expectancy (the age to which they live with a probability of 50%) will not change significantly.

Physical interpretation

Thanks to modern statistical studies of mortality emerged and developed many mathematical theories, in particular models of the decay of atoms. The object is considered anti-aging, if the probability of the existence of an object within a certain time interval does not depend on the time that the object already existed. It is obvious that man does not possess this property, because the older she is, the more likely she will die within a given period of time. For example, radioactive atoms are anti-aging.

If the average length of anti-aging existence of the object is equal to T, then the probability that he will not cease his existence for the next period of time x, is equal to $e^{-x/T}$, where x is a positive number. Besova property of radioactive particles stems from the fact that the decay rate is proportional to the number of particles that are not disintegrated. The coefficient of proportionality is called the decay constant and denoted by λ. If at time $t = 0$ was N_0 particles that have not broken up (because the rate of decay is constant, which is obtained by integration) at time x the number of particles that are not disintegrated, is $N_x = N_0 e^{-\lambda x}$. This means that the probability of survival to time x equal to $e^{-\lambda x}$. Accordingly, radioactive particles really besko, and their average lifetime is equal to a mathematical spoden (from time of "living" particles) $M(x) = T = 1/$. In other words, the lifetime of the radioactive particles described in index distribution with parameter λ, i.e. has the probability density is $e^{-\lambda x}$. The half-life anti-aging object (the period during which half of the objects ceases to exist) is the root of the following equation: $e^{-\lambda x} = \frac{1}{2}$,, i.e. $x = \ln 2\ /$, this value is the median of the distribution.

The reasons for this paradox is that the expectation, generally speaking, does not coincide with the median; by changing statistical data, the median may remain unchanged, but the expectation is changing.

This paradox it is useful to consider when studying the topic "distribution function, density function and numerical characteristics of continuous random variables", namely during the study of exponential distribution. After the proposed paradox is expedient to solve this problem.

Task. To determine the half-life of radioactive substances, if the rate of decay of the nucleus of any atom per unit time is constant and equal to p. (Half-time T_n is defined by the moment when the mass of a radioactive substance is reduced by half) [5].

The paradox de Muavre and energy saving [6]

Let is $2n$ tosses of a symmetric coin. To aspire to the probability that the number of deletions of the coat of arms will be equal to the number of deletions number as $n \to \infty$?

According to the law of large numbers Bernoulli, is the probability that when flipping a coin the number of times the emblem is approximately equal to the number of drop number, if you increase the number of tosses tends to 1.

It turns out that the probability that the number of arms is equal to the number of numbers that tends to zero. For example, if 6 toss coins the probability of getting 3 coats is equal to 5/16, 100 toss the probability of occurrence is equal to 50 coats of arms of 0.08; with 1000 toss the probability of getting 500 emblems is less than 0.02. In general, if you toss a coin $2n$ times, then the probability that the emblem will fall exactly n times, is equal to $p = C_{2n}^{n} 2^{-2n}$, and for sufficiently large n the probability p is approximately equal to $1/\sqrt{\pi n}$, indeed tends to zero if n tends to infinity. To sum up the above: the probability that the number of arms is approximately equal to the number of numbers tends to 1, while the probability that the number of arms equal to the number of numbers that tends to 0. The gap between these two facts was surrounded by an "atmosphere of mind" until then, while de Muavre had built over them the mathematical bridge.

The explanation for the paradox

Let H_n and $T_n$determine the number of appearance of the coat of arms and numbers, respectively, when n toss coins. According to the law of large numbers Bernoulli, is the probability that the difference H_n- T_n becomes very small compared to n, tends to 1 (which is natural). However, de Muavre noticed that the size $|H_n - T_n|$is not rather small compared to the magnitude of $\sqrt{n}$. He calculated, for example, that for $n = 3600$is the probability that $|H_n - T_n|$ does not exceed 60 0... 682688 . Let x be an arbitrary positive integer and let $A_n(x)$ denotes the probability that $|H_n - T_n| < x\sqrt{n}$. De according to de Muavre, $A_n(x)$ with increasing n approaching the value $A\ \ (x)$ that is between 0 and 1. With increasing x from zero to one $A\ \ (x)$ is constantly increasing from zero to one. This function $A\ \ (x)$ is the mathematical bridge, which was discussed above. To find $A\ \ (x)$ de Muavre used the Stirling formula.

Physical interpretation

Limit theorem, de Muavre-Laplace can be used in a variety of industries when planning, for example, energy consumption.

Example. Consider 200 similar machines in the factory. If on average 80% of the machines are working and 20% are in repair, it is necessary to provide energy for 160 machines. However, sometimes it can work and all 200 machines. How much energy is needed to provide the factory with a probability of 99.9% of all the working machines to work? (This refers to the fact that machines fail independently of each other.)

In the above formula $H_n$now means the number of working machines, $n = 200$ and $p = 0{,}8$. According to the table of the standard normal distribution function for $x = 2{,}98$ find $\Phi(x) \approx 0{,}999$. These values are $np + x\sqrt{np(1-p)} = 160 + 3\sqrt{32}$, so it is enough to consider 177 machines. (In practice, however, account for almost all 200 machines, thus showing an excess of caution.)

This paradox is worth exploring the theme of "Limit theorems of probability theory". You should pay attention to the fact that students need to recall the Stirling formula.

The paradox of Brownian motion [6]

The trajectory of Brownian motion is quite irregular (nowhere differential). Usually any irregular curve such as the trajectory of Brownian motion in the plane, considered as one-dimensional. It can be shown that the trajectory of Brownian motion on the plane actually fills the whole plane (in an arbitrarily given neighborhood of an arbitrary point of the trajectory falls with the probability of 1). It follows that the trajectory can be considered as two-dimensional curves. Which of these approaches to use?

The explanation for the paradox

The notion of dimension in day-to-day value was used in the beginning of the century. Curves, surfaces and body were seen as one-dimensional, two-dimensional and three-dimensional objects. Usually they say that a figure has dimension k, if for "profiling" but you need k parameters (coordinates). Using intuitive ideas of Poincare, Brouwer in 1913 defined the topological dimension. Later, in 1922, Menger and Uryson, working independently, came to this concept. In the definition of the topological dimension of Brownian motion is one-dimensional. On the other hand, in 1919 Hausdorff introduced the notion of dimension, according to which Brownian motion is two-dimensional. In d -dimensional Euclidean space the volume of a unit ball is equal to $v(d) = \frac{\Gamma^d(\frac{1}{2})}{2^d\Gamma\left(1+\frac{d}{2}\right)}$, where Γ denotes the gamma function. This expression makes sense for fractional $d \geq 0$. Let n - dimensional Euclidean space is taken as the set E, which is covered by nitely many n - dimensional balls with radii $r_1, r_2, \dots r_k$. Then d is the Hausdorff measure of the set E equal

$$\lim_{r\to 0} inf_{r_i \leq r} \sum_{i=1}^{k} v_i(d) r_i^d$$

Bezykovych proved that there always exists a (real) number D if $d < D$, then d is the measure of the set E is infinite, but in the case of $d > D$ it is equal to 0. This number D is called the Hausdorff dimension or Hausdorff-Bezykovych the set E. If it were so, the dimensions are not necessarily integer. For example, both coordinates of the Brownian motion on the plane as a function of time (i.e., the curves "a one-dimensional Brownian motion") have Hausdorff dimension 3/2. Respectively these curves are somewhere between the "real" curves and "real" surfaces. The dimension of the Brownian motion curve in the plane equal to 2, as in the "real" surfaces.

This paradox should be addressed in the classroom with students in physics, enrolled in master's degree because undergraduate students do not possess adequate concepts of n-dimensional Euclidean space, measures and dimensions. Also paradox can offer students who are studying on specialty "Mathematics (additional major: physics)".

Conclusions. It is advisable to use methodologically important counterexamples, paradoxes and their physical interpretations in the study of probability theory and mathematical statistics will allow:

1) to increase learning efficiency of physics students;

2) to focus students ' attention on the limits of applicability of certain concepts and laws of probability theory and mathematical statistics;

3) establish interdisciplinary connections with theoretical physics;

4) to demonstrate the common and distinctive features of mathematical and physical models and models and their interpretations;

5) to contribute to a deeper understanding of the methods of probability theory and their applications.

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