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Model-based parallel construction of a numerical-analytical scheme

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Abstract: The aim of this work is to construct a numerical-analytical method of designing efficient algorithms for solution of tasks having the parabolic type. Using a priori information about the smoothness of solutions, great attention is paid to the construction of solutions of high -order accuracy.

Keywords: numerical-analytical method, parabolic type, solutions of high-order accuracy.

Creation of parallel computing systems required the development of mathematical concepts for constructing parallel algorithms, i.e. algorithms adapted for implementation in these systems. As the basis for constructing the parallel algorithm we can take both: a sequential algorithm and the task itself as well [2, 3]. The most sensible at parallelization of sequential algorithm is pragmatic approach; actually sequential algorithms detect common elements which further are transformed to a parallel form.

The numerical and analytical schemes consider the example of the boundary value problem for the heat equation with constant coefficients. E.g. we want to find a solution in the area $\{0 \le x < x_L,\ 0 < t \le T\}$ [4]:

$$\frac{\partial Y}{\partial t} = \frac{\partial^2 Y}{\partial x^2}, \qquad (1)$$

which satisfies the initial condition

$$Y(x,o) = \varphi(x) \qquad (2)$$

and the boundary conditions

$$Y(0,t) = \mu_0(t), \qquad Y(x_L,t) = \mu_L(t) \quad (3)$$

Here $\mu_0(t),\ \mu_L(t),\ \varphi(x)$ – are given functions. It is known that under certain assumptions of the smoothness problem (1) - (3) has a unique solution [4].

We propose to apply the net on value x with the step between nodes equal to.

$$Dx1_p = \frac{x_L}{2m}, \quad p = \overline{1, 2m-1}, \quad m \in Z, \quad (4)$$

where m - is the integer parameter sampling. For uniformly distributed nodes

$$\begin{cases} Dx1 = x_p - x_{p-1} = const, \\ x_p = x_{p-1} + p \cdot Dx1, \quad p = \overline{1, 2m-1}. \end{cases} \qquad (5)$$

On the basis of prior information required function is represented as a Taylor series:

$$Y_{p+\varepsilon_x,1}(t,x) = \sum_{n=0}^{\infty} \varepsilon_x^n \cdot Y_{p,n+1}(t), \qquad (6)$$

with

$$\begin{cases} \varepsilon_x = \dfrac{x - x_p}{x_{p+1} - x_p} \in [+1, -1], \\ Y_{p,n+1} = \dfrac{Dx1^n}{n!} \cdot \left.\dfrac{\partial Y}{\partial x^n}\right|_{x=x_p}. \end{cases}$$

After agreement (6) with equation (1) and equating the coefficients of equal powers we receive ε_x^n, the system of ordinary differential equations (SODE)

$$Y'_{p,n+1}(t) = \frac{(n+1)(n+2)}{Dx1^2} \cdot Y_{p,n+3}(t) \qquad (7)$$

having the form of Cauchy

$$Y_{p,n+1}(0) = \varphi_{p,n+1}, \qquad (8)$$

Where $\varphi_{p,n+1}$ – are the known values of the Taylor component of the initial function (2).

Let restrict a finite number of terms $n = N$ series in the right side of the Taylor series (6), so we obtain

$$Y_{p+\varepsilon_x,1}(x,t) = \sum_{n=0}^{N} \varepsilon_x^n \cdot Y_{p,n+1}(t), \qquad (9)$$

Where N – is the integer number .To approximate equation (1) in the point (x_p, t) we will consider the closing connection

$$\begin{Bmatrix} Y_{p,N+1} \\ Y_{p,N} \end{Bmatrix} \qquad (10)$$

We suppose that in (9) $\varepsilon_x = \pm 1$ and thus we obtain on the three-point template the system of two algebraic equations

$$\begin{cases} Y_{p,N+1} + Y_{p,N} = \left[Y_{p+1,1} - \sum_{n=0}^{N-2} Y_{p,n+1} \right], \\ Y_{p,N-1} - Y_{p,N-1} = (-1)^N \cdot \left[Y_{p-1,1} - \sum_{n=0}^{N-2} (-1)^n \cdot Y_{p,n+1} \right]. \end{cases} \qquad (11)$$

We find

$$\begin{Bmatrix} Y_{p,N+1} \\ Y_{p,N} \end{Bmatrix} = \frac{1}{2} \cdot \left\{ \left[Y_{p+1,1} \pm (-1)^N \cdot Y_{p-1,1} \right] - \sum \varphi_n^{\pm} \cdot Y_{p,n+1} \right\}, \qquad (12)$$

Where

$$\varphi_n^{\pm} = 1 + (-1)^{n+N}, \quad N = 2, 3, 4, \ldots \qquad (13)$$

are normalizing factors.

For N = 2 $n = \overline{0, 0}$ we have

$$\begin{cases} Y_{p,2} = \frac{1}{2} \cdot \left[Y_{p+1,1} - Y_{p-1,1} \right], \\ Y_{p,3} = \frac{1}{2} \cdot \left\{ \left[Y_{p+1,1} + Y_{p-1,1} \right] - 2 \cdot Y_{p,1} \right\}. \end{cases} \qquad (14)$$

After substituting (14) into (7) we obtain the SODE

$$Y'_{p,1}(t) = \frac{1}{Dx1^2} \cdot \left\{ \left[Y_{p+1,1}(t) + Y_{p-1,1}(t) \right] - 2Y_{p,1}(t) \right\}, \qquad (15)$$

$p = \overline{1, 2m-1},$

Where $\left\{ Y_{0,1}(t), Y_{2m,1}(t) \right\}$ are the boundary functions of the first kind

For N = 3 and the significance of the relations (7) and (14) we obtain the higher-order SODE

$$\begin{cases} Y'_{p,1}(\tau) = \frac{1}{2 \cdot Dx1^2} \cdot \left[Y_{p+1,2}(\tau) - Y_{p-1,2}(\tau) \right], \\ Y'_{p,2}(\tau) = \frac{1}{Dx1^2} \cdot \left[Y_{p+1,2}(\tau) + Y_{p-1,2}(\tau) - 2Y_{p,2}(\tau) \right], \end{cases} \qquad (16)$$

where

$$\begin{cases} Y_{0,2}(\tau) = Dx1 \cdot gW(\tau), \\ Y_{2m,2}(\tau) = Dx1 \cdot gL(\tau), \end{cases} \qquad (17)$$

are known boundary functions of the second kind.

Note that the developed approach includes conventional finite-difference methods in a special case. Scheme (15) coincides with the classical Dirichlet problem, and the circuit (16) with the Neumann problem. The problem (16) is characterized by the fact that the transmission of information on the boundaries of the area in the natural scheme is implemented through internal point accurately without reducing the order of approximation.

With the increase of N - order reducing the approximation orders of closing bonds (12) also increases. Note that the integration of SODE (15) - (17) having the Cauchy form with explicit methods is the most advanced procedure. The variety of standard programs allows us to consider this process as an elementary. From the point of view of cost effectiveness depending on operations' number for the mentioned above methods cannot be improved.

The developed numerical and analytical procedure for discretization can be simply generalized to other types of differential equations of mathematical physics. In particular, in the stationary problems it is easier to localize features in the regions of smoothness using schemes of high order accuracy.

The value of the order of approximation in conjunction with carrying out the calculation on the shredder grids allows to focus in assessing the calculation accuracy.

We will show how to formulate the algorithm of approximate calculations based on the operations with functions as well as with formulas.

In the construction of a computational algorithm (13) - (17) we used a priori information available to the task, and first of all information about belonging to a particular class of functions' smoothness which describe the task. Smoothness is determining feature of the diameters' size .The values of the diameters give an idea of the best possible accuracy for the computational algorithm [1].

Let us introduce Cauchy data as dependent variables

$$\left\{ Y_{p,1}(t),\ Y_{p,2}(t) \right\},\ p = \overline{1, 2m-1}. \qquad (18)$$

Rewriting SODE (4) as follows

$$Y_{p,n+3}(t) = \frac{Dx1^{2n}}{(n+1)(n+2)} Y'_{p,n+1}(t). \qquad (19)$$

From (18), (19) we receive

$$\begin{cases} Y_{p,3}(t) = \frac{Dx1^2}{2!} Y^{(1)}_{p,1}(t), \\ Y_{p,4}(t) = \frac{Dx1^2}{3!} Y^{(1)}_{p,2}(t), \\ Y_{p,5}(t) = \frac{Dx1^4}{4!} Y^{(1)}_{p,1}(t), \\ Y_{p,6}(t) = \frac{Dx1^6}{6!} Y^{(1)}_{p,2}(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad (20)$$

Thus, the general solution of (6) can be represented as follows

$$Y_{p+\varepsilon_x,1}(t,x) = \sum_{n=0}^{\infty} \varepsilon_x^{2n} \cdot \frac{Dx1^{2n}}{(2n)!} \cdot Y^{(n)}{}_{p,1}(t) + \sum_{n=0}^{\infty} \varepsilon_x^{2n+1} \cdot \frac{Dx1^{2n+1}}{(2n+1)!} \cdot Y^{(n)}{}_{p,2}(t). \qquad (21)$$

The first term of (21) satisfies the adiabatic wall, and the second one satisfies the conditions of the wall at a constant temperature.

In the algebraic area the mathematical model in the form of Cauchy data duplexing

$$\sum_{n=0}^{\infty} Y^{(n)}{}_{p,1}(t) \cdot \frac{Dx1^{2n}}{(2n)!} = \frac{1}{2} \left[Y_{p+1,1}(t) + Y_{p-1,1}(t) \right], \qquad (22)$$

$$\sum_{n=0}^{\infty} Y^{(n)}{}_{p,2}(t) \cdot \frac{Dx1^{2n+1}}{(2n+1)!} = \frac{1}{2} \left[Y_{p+1,2}(t) + Y_{p-1,2}(t) \right] \qquad (23)$$

In general we have the rapid convergence of infinite series (22), (23) at physically realizable of the variables. For example, if the derivatives of $Y_{p,1}(t)$, $Y_{p,2}(t)$ are limited by derivatives of the exponential functions, this is a confirming of term

by term differentiation which is used in the analysis. However, in practical cases, the series must converge quickly enough to be able to confine to a few initial terms of the series.

Example. Let us consider a solid plate with a stepped increase in the surface temperature:

$$\begin{cases} Y_{0,4}(t)=1,\ Y_{2m,1}(t)=1,\ x_{2m}=2, \\ Y_{m,1}(t)-\text{average surface} \\ \varepsilon_x\in[-1,\ +1]. \end{cases} \qquad (24)$$

Taking in account that $\varepsilon_x\big|_{x=0}$ and the symmetry of the problem at coordinate, we have $Y_{m,2}=0,\ Dx1=1$. To define $Y_{m,1}$ we use outer boundary conditions on the surface $\varepsilon_x=\pm 1$. From the infinite sum in (21) we leave only two first terms at n = 1 and n = 2 of the series. So we get the mathematical model

$$\begin{cases} N=1,\ Y_{m,1}(t)+\dfrac{1}{2}Y'_{m,1}(t)=1, \\ N=2,\ Y_{m,1}(t)+\dfrac{1}{2}Y'_{m,1}(t)+\dfrac{1}{4}Y''_{m,1}(t)=1. \end{cases} \qquad (25)$$

From which we receive the result:

$$\begin{aligned} &N=1, \\ &Y_{m,1}(t)=1-e^{-2t}; \\ &N=2, \\ &Y_{m,1}(t)=1-1{,}37\,e^{-2{,}54t}+1{,}37\,e^{-9{,}46t}; \\ &N=\infty, \\ &Y_{m,1}(t)=1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}e^{-(2n-1)^2\frac{\pi^2}{4}t}. \end{aligned} \qquad (26)$$

When passing to the limit at $N\to\infty$ approximate solution matches to the exact one. When $N=2$ the computing error does not exceed 3 %. Thus, with increasing parameter N the error decreases rapidly (see Table 1).

Table 1.

Temperature rise on the inner surface of the plate on the outer surface

t	N=1	N=2	The exact solution
0	0	0	0
0.05	0.095	0.026	0.03
0.10	0.181	0.082	0.081
0.20	0.330	0.238	0.230
0.40	0.551	0.499	0.528
0.65	0.698	0.413	0.711
0.80	0.798	0.819	0.827
1.00	0.864	0.892	0.892

As the aim was to synthesize parallel algorithms of the method with the help of ratio (26) we obtained that the method fits into the concept of unlimited parallelism [2]. Indeed, one processor can be assigned to one node of the design, and it becomes possible to perform calculations on all nodes simultaneously.

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Модели параллельных вычислений на основе конструирования численно-аналитических схем

Аннотация: Работа посвящена построению численно-аналитическим методом конструирования эффективных алгоритмов для решения задач параболического типа. Используя априорную информацию о гладкости решения, большое внимание уделяется построению решений высокого порядка точности.

Ключевые слова: численно-аналитические методы, задач параболического типа, высокого порядка точности.