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Optimum blade profile of a groundthrower as a solution of the brachistochrone for the centrifugal force of inetia

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Abstract. It has been described a method for finding the brachistochrone for the centrifugal force of inertia. A time functional in the polar coordinate system has been built and the corresponding Euler equation has been recorded; its first integral has been obtained, an analytic solution for the integral has been found. It has been ascertained that its structure depends on the boundary conditions. Examples of calculations of optimal trajectories have been provided; the blade of the groundthrower with brachistochrone profile has been built.

Keywords: brachistochrone, the centrifugal force of inertia, functional, polar coordinates, the Euler equation, the optimum blade profile.

Introduction. When creating some technical (technological) devices there is the problem of choosing an optimal form of guides in which some particles move (e.g. ground ones) under pressure of centrifugal forces of inertia. In particular, the actual problem is the creation of groundthrower mechanisms to be used for extinguishing fires in areas where there is a lack of water sources. Similar mechanisms are used in road, agricultural machinery etc. Rotary groundthrowers have a number of advantages as they throw ground using blades located on the rotating rotor. Methods for selecting form, position of blades, analysis of motion of soil particles on them are currently not enough developed. Researches on the issues resolution that are aimed at improve technological characteristics of the devices are relevant.

The task of choosing the optimal blade profile can be schematized as a problem of determining the shape of a curve in the field of centrifugal forces of inertia, which provides the minimum motion time of a particle (the problem of the brachistochrone for the centrifugal force).

It is known that the classical problem of the brachistochrone for uniform field strength of gravity was the starting point for the creation of the calculus of variations [3]. Salvations of similar problems for some types of centrifugal forces are given in [1]. Questions about constructing an optimal trajectory of the point in the field of centrifugal forces of inertia are reviewed in an article [4].

In this paper, we propose a method for solving the problem of the brachistochrone for a central force, which is the centrifugal force of inertia. Functional time built in the polar coordinate system. For the corresponding Euler equation obtained his first integral in the form of first order differential equation. An analytical solution of this equation. It has been established that its structure depends on the boundary conditions. Results of calculations of optimal trajectories constructed blade profile brachistochrone.

1. Building the time functional. Let the points *A* and *B* are located in the central field of repulsive forces – centrifugal forces of inertia. Let's draw a plane through the points *A* and *B* and the center repulsion point *O* (Fig. 1). Consider the curves joining these points located on the plane. We choose from these curves such that a material point *A* at a speed $v_0 = 0$ of opening by moving only un-

der the influence of centrifugal force reaches the point *B* in the minimum time. Solution of the problem and the conclusion of the functional movement time conveniently carried out in the polar coordinate system centered at the specified point. The current coordinates of the point *M* are denoted as как ρ and φ; the coordinates of points, *A* and *B* – respectively (ρ_0, φ_0) and (ρ_1, φ_1).

The projection of the centrifugal force of inertia of the material point associated with the rotating body, the direction of the radius ρ has the form

$$F_\rho = ma_\omega = m\omega^2\rho \quad (1)$$

where *m* – the mass of the point; $a_\omega = \omega^2\rho$ – axis-directed (normal) acceleration; ω –angular velocity.

Then the expression for the potential energy of the centrifugal force of inertia can be represented as [2]:
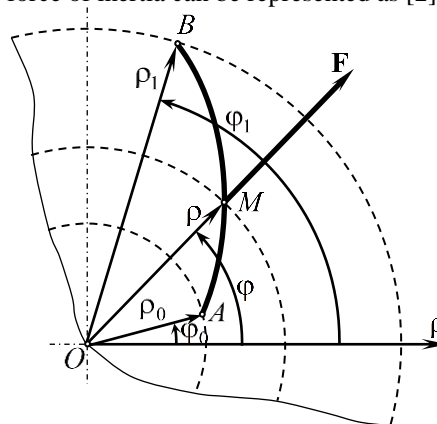

Fig. 1. Scheme for construction brachisto-chrone in a centrifugal force field

$$\Pi = \int_\rho^0 F_\rho d\rho = -\frac{m\omega^2}{2}\rho^2 . \quad (2)$$

When writing the formula (2) it was assumed that the initial position to determine the potential energy is a repulsive center, in which the potential energy is zero.

In a coordinate system uniformly rotating around a fixed axis, if we neglect the forces of friction and re-

sistance, the law of conservation of energy in a relative motion [2]

$$T + \Pi = h = const, (3)$$

where $T = \dfrac{1}{2}mv^2$ – kinetic energy of a point in the relative motion with respect to the rotating coordinate system;

$$h = -\dfrac{m\omega^2}{2}\rho_0^2 - \text{constant energy.}$$

In the future, we believe that in the formula (3) it takes into account only the potential energy of the centrifugal force of inertia. From the point of view of applications, that is the most important case. An angular velocity in technological devices is such that the effect of gravitational forces on the motion of a point slightly. In the analysis of movement on the horizontal plane of the reservation is generally unnecessary.

Now on the basis of (3) for the velocity point, we have

$$v = \omega\sqrt{\rho^2 - \rho_0^2} \quad (\rho \geq \rho_0). (4)$$

From the definition of the algebraic value of velocity (a projection of velocity on the tangent to the trajectory) $v = v_\tau = \dfrac{ds}{dt}$, the expression of the differential for the square of the arc in polar coordinates $ds^2 = d\rho^2 + \rho^2 d\varphi^2$ and the formula (4) it follows that

$$dt = \dfrac{ds}{v} = \dfrac{\sqrt{\rho'^2 + \rho^2}}{\omega\sqrt{\rho^2 - \rho_0^2}}\, d\varphi, (5)$$

where $\rho' = \dfrac{d\rho}{d\varphi}$.

Note that when counting the arc in the direction of motion of the point differential path will coincide with the differential of the arc coordinate ds and the point velocity module will coincide with its algebraic value $v = v_\tau$.

Integrating, we are obtaining the functional

$$\tau[\rho(\varphi)] = \dfrac{1}{\omega}\int_{\varphi_0}^{\varphi_1} \dfrac{\sqrt{\rho'^2 + \rho^2}}{\sqrt{\rho^2 - \rho_0^2}}\, d\varphi. (6)$$

2. Search functional extremum. For the integrand we introduce the notation

$$P = P(\rho, \rho') = \dfrac{\sqrt{\rho'^2 + \rho^2}}{\sqrt{\rho^2 - \rho_0^2}}, (7)$$

then on the curve realizing extremum of the considered functional, the condition must be satisfied (this follows from the necessary conditions for an extremum of the functional (6) [3])

$$P_\rho - \dfrac{d}{d\varphi}P_{\rho'} \equiv 0, (8)$$

where P_ρ, $P_{\rho'}$ – derivatives P respectively ρ and ρ'.

Thus, the desired function $\rho = \rho(\varphi)$ is the solution of second order differential equation (Euler equation) (8) or in expanded form

$$P_\rho - P_{\rho\rho'}\rho' - P_{\rho'\rho'}\rho'' = 0, (9)$$

where $\rho'' = \dfrac{d^2\rho}{d\varphi^2}$.

After multiplying this equation term by term on its left-hand side becomes an exact derivative

$$\dfrac{d}{d\varphi}\left(P - \rho' P_{\rho'}\right).$$

Consequently, the Euler equation has the first integral

$$P - \rho' P_{\rho'} = \dfrac{1}{C}. (10)$$

From expression (10), after transformations, we obtain

$$\dfrac{d\rho}{d\varphi} = \rho\sqrt{\dfrac{C^2\rho^2}{(\rho^2 - \rho_0^2)} - 1}. (11)$$

Differential equation (11) admits an analytic solution (cumbersome calculations are not presented here)

$$\varphi = \begin{cases} \operatorname{arctg} z + \dfrac{1}{z} + C_1, & C^2 = 1, \\[2mm] \operatorname{arctg} z - \dfrac{1}{\sqrt{1-C^2}}\operatorname{arctg}\dfrac{z}{\sqrt{1-C^2}} + C_1, & C^2 < 1, \\[2mm] \operatorname{arctg} z - \dfrac{1}{2\sqrt{C^2-1}}\ln\left|\dfrac{z - \sqrt{C^2-1}}{z + \sqrt{C^2-1}}\right| + C_1, & C^2 > 1, \end{cases} \quad (12)$$

where $z = \sqrt{\dfrac{C^2\rho^2}{\rho^2 - \rho_0^2} - 1}$. (13)

Boundary conditions for finding of permanent C and C_1 taking into account (13):

as $\varphi = \varphi_0$, $\rho = \rho_0$, $z = z(\rho_0) = z_0 = \sqrt{\dfrac{C^2\rho_0^2}{\rho_0^2 - \rho_0^2} - 1} = \infty$; (14)

as $\varphi = \varphi_1$, $\rho = \rho_1$, $z = z(\rho_1) = z_1 = \sqrt{\dfrac{C^2\rho_1^2}{\rho_1^2 - \rho_0^2} - 1}$. (15)

The greatest interest for practice is in the cases where $C^2 < 1$ and $C^2 > 1$. Here we are considering the conditions (14), (15) and constructing the transcendental equation for the C^2 in the first case

$$f(x) = \varphi_1 - \operatorname{arctg}\sqrt{\dfrac{x\rho_1^2}{\rho_1^2 - \rho_0^2} - 1} + \dfrac{1}{\sqrt{1-x}}\operatorname{arctg}\dfrac{\sqrt{\dfrac{x\rho_1^2}{\rho_1^2 - \rho_0^2} - 1}}{\sqrt{1-x}} - \varphi_0 + \left(1 - \dfrac{1}{\sqrt{1-x}}\right)\dfrac{\pi}{2} = 0 \quad (16)$$

where $x = C^2$.

After that C_1 is determined by the boundary conditions provided

$$C_1 = \varphi_0 - \left(1 - \dfrac{1}{\sqrt{1-C^2}}\right)\dfrac{\pi}{2}. (17)$$

Point values of the constants C and C_1 using the second formula in (12) and the formula (13) allow us to write the following expression for the unknown function

$$\varphi(\rho) = \operatorname{arctg}\sqrt{\frac{C^2\rho^2}{\rho^2-\rho_0^2}-1} - \frac{1}{\sqrt{1-C^2}}\operatorname{arctg}\frac{\sqrt{\dfrac{C^2\rho^2}{\rho^2-\rho_0^2}-1}}{\sqrt{1-C^2}} + C_1. \quad (18)$$

Examples constructed using the method developed brachistochrone are shown in Fig. 2 ($\rho_0 = 0{,}496\text{м}$, $\rho_1 = 0{,}632\text{м}$, $\varphi_0 = 0$, $\varphi_1 = 20^\circ$). Fig. 2a shows, obtained by (18), a schedule convenient for analysis of the inverse function $\rho = \rho(\varphi)$. Fig. 2b this dependence graphs are constructed in the polar coordinate system for angles φ_1 multiple to 20°, and for ρ_0 there were undertaken the minimum limit values ρ_0^*, which still provide

the opportunity to build solutions using the second formula (12) ($C^2 < 1$).

3. Formation of the blade profile brachistochrone. Fig. 3 shows groundthrower scheme: 1 – hub, 2 – ring, 3 – spoke 4 – curved blade. It is assumed that the thrower rotates with angular velocity ω upstream clockwise. And radii R_1 and R_2 are the radii of a circle passing through the back and the front edge of the blade. In the above theory, they correspond to the polar radii ρ_0 and ρ_1. The curved blade with a profile in the form of brachistochrone (see Fig. 2) is shown in Fig. 4. When forming the side walls of said blade also used brachistochrone. Protruding portions of the side walls can serve as a kind of disintegrating agents, facilitating the introduction of the blade into the ground.
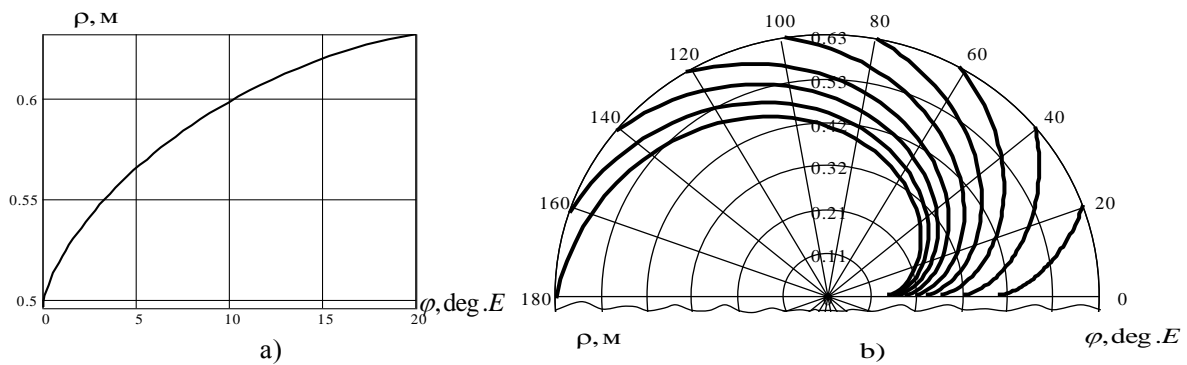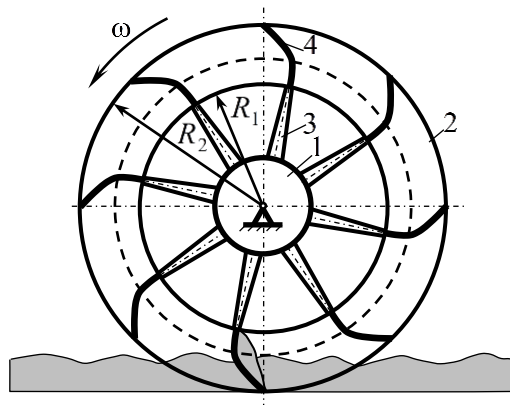

Fig. 2. Function graphics $\rho(\varphi)$ ($\varphi_0 = 0$, $\rho_1 = 0{,}632\text{м}$):

a) – $\rho_0 = 0{,}496\text{м}$, $\varphi_1 = 20^\circ$; b) – ρ_0^* – the minimum limit values

Research results allowed us to establish a number of fundamental advantages curved blade connected with the fact that such a blade embedded in the ground at a more acute angle than straightforward: a larger volume of the captured ground, less effort in the implementation of smaller dynamic loads on the blades and rotor, lower power of drive motor.

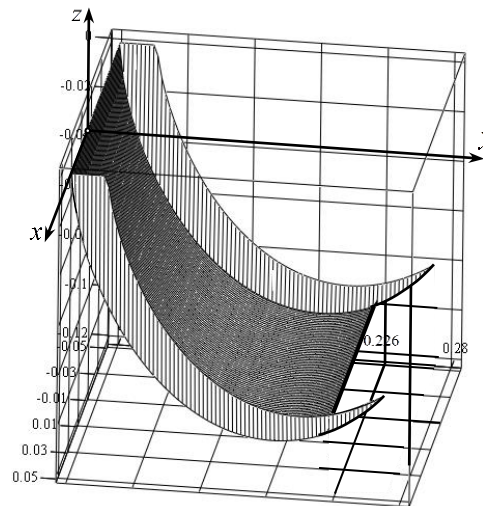**Fig. 3.** Scheme of the groundthrower mechanism

Fig. 4. Blade with brachistochrone profile

Conclusion

1. It has been developed a method for solving the problem of the brachistochrone for the centrifugal force of inertia.

2. It has been built the time functional in the polar coordinate system.

3. It has been obtained the first integral of the Euler equation in the form of first order differential equation and found its analytical solution.

4. It has been he determined the dependence of the mathematical description of the optimal trajectories on the values coorinates of the starting and ending points.

5. The results of calculations of optimal trajectories have been displayed.

6. Built groundthrower blade with brachistochrone profile has been built.

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Шатохин В.М., Семкив О.М., Попова А.Н. Оптимальный профиль лопатки грунтометателя как решение задачи о брахистохроне для центробежной силы инерции

Аннотация. Изложен метод нахождения брахистохроны для центробежной силы инерции. Построен функционал времени в полярной системе координат и записано соответствующее уравнение Эйлера; получен его первый интеграл, для которого найдено аналитическое решение. Установлено, что его структура зависит от краевых условий. Приведены примеры расчетов оптимальных траекторий, построена лопатка грунтометателя с профилем брахистохроны.

Ключевые слова: брахистохрона, центробежная сила инерции, функционал, полярные координаты, уравнение Эйлера, оптимальный профиль лопатки.