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Modeling of minimal surfaces based on isotropic curves and quasiconformal change of parameter

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Abstract. The paper is proposed constructing of minimal surfaces based on spatial isotropic Bezier curves. Modeling of isotropic curves is realized in a complex space. Several methods of constructing such curves: based on analytic functions of a complex variable, based on a specified real curve in the plane and based on the deformation of a plane curve in the complex space are proposed. Conditions of isotropy of the Bezier curve of the n^{th} order are found and the curve of the 3^d order is constructed as an example. A special case of constructing a Bezier curve based on isotropic sides of the characteristic quadrilateral is examined, in this case the real curve will be on the plane in three-dimensional space. The resulting curves were used as a guide curve for constructing surfaces. To create a surface the quasiconformal change of parameter in the equation of an isotropic curve is done. By separating the real and imaginary parts we get the equations of the surfaces. The analysis of the coefficients of the first and second quadratic forms is done. It was proved that the resulting surface are minimal surfaces.

Keywords: *isotropic curves, Bezier curves, minimal surfaces*

Problem statement. Minimal surfaces are widely distributed in the nature as the most economical surfaces: they have the smallest space between the closing them contour, tensions in all directions of the surface are the same. These properties of minimal surfaces determine their advantages in design and construction. There is a class of problems where their solution depends on functional and esthetic needs. In addition to the quantitative characteristics it is necessary to consider the practical experience of the developer. Bezier method has been developed to work in the interactive mode and for intuitive representation of the influence of tangent on the curve shape. Due to the new opportunities that have emerged due to the development of computer technology, the problem of constructing unlimited minimal surfaces based on the guide isotropic Bezier curve has been emerged.

Analysis of publications. One of the methods of constructing minimal surfaces were offered by Weierstrass [1], he was constructing these kinds of surfaces on the basis of isotropic curves or minimal curves which length in the complex space is equal to zero:

$$x(t)'^2 + y(t)'^2 + z(t)'^2 = 0\,. \qquad (1)$$

The theory of minimal curves (isotropic) was founded by Sophus Lie [4]. Isotropic geometry was developed by Strubecker in the 1940s. The main states are in the monograph [2]. Kartan and his followers were constructed and researched the isotropic curves in the complex space [3,4], the main focus was on constructing the isotropic curves on the basis of the moving reference points. Weierstrass proposed a non-quadrature representation of isotropic curves [1]. Separately, you can highlight the works of applied geometry [5].

The purpose of this work is development of the method of constructing minimal surfaces based on the isotropic Bezier curves of the n^{th} order. Using Bezier curves will make the method of generating of surfaces flexible to changes of formation conditions of the guide curve.

The basic part. Bezier curve will have the next form:

$$\mathbf{r(t)} = \sum_{j=0}^{n} \mathbf{r_j} J_{n,j}(t), \text{ де } J_{n,j}(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}\,, \qquad (2)$$

where $\mathbf{r_j} = \lfloor x_j \quad y_j \quad z_j \rfloor$

- Spatial isotropic Bezier curves of the n^{th} order based on analytic functions

Let's construct isotropic curve of the n^{th} order on the basis of analytical function of Weierstrass method.Let the function $f(t)$ is defined as: $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n$, where $a_0, a_1, a_2, a_3, \ldots a_n$ - some complex values. Equation of isotropic curve will have the form [1]:

$$\begin{aligned} x &= i \cdot \left\{ \left(f(t) - t \cdot f'(t) - \frac{1-t^2}{2} f''(t) \right) \right\}, \\ y &= \left\{ f(t) - t \cdot f'(t) + \frac{1+t^2}{2} f''(t) \right\}, \\ z &= -i \cdot \left\{ f'(t) - t \cdot f''(t) \right\}. \end{aligned} \qquad (3)$$

Let's find derivatives $f'(t)$ and $f''(t)$ and let's substitute into the equation of isotropic curve (3), we will have:

$$\begin{aligned} x &= i \cdot \{ \sum_{j=0}^{n-2} \left[\frac{(j-1)(j-2)}{2} a_j - \frac{(j+1)(j+2)}{2} a_{j+2} \right] t^j + \sum_{j=n-1}^{n} \frac{(j-1)(j-2)}{2} a_j t^j \}, \\ y &= \{ \sum_{j=0}^{n-2} \left[\frac{(j-1)(j-2)}{2} a_j + \frac{(j+1)(j+2)}{2} a_{j+2} \right] t^j + \sum_{j=n-1}^{n} \frac{(j-1)(j-2)}{2} a_j t^j \}, \\ z &= i \cdot \left\{ \sum_{j=0}^{n-1} (j+1)(j-1) a_{j+1} t^j \right\}. \end{aligned} \qquad (4)$$

To go from equation (4) to the form (2) it is necessary to find the value of the radius vector of the characteristic points of the polygon $\mathbf{r_j} = \lfloor x_j \quad y_j \quad z_j \rfloor$. To do this we have to compare the coefficients of equations (2) and (4) of the appropriate t^j. We obtain n equations for each coordinate:

If $0 \le j \le n-2$

$$i \cdot \left\{ \frac{(j-1)(j-2)}{2} a_j - \frac{(j+1)(j+2)}{2} a_{j+2} \right\} = \sum_{l=0}^{j} (-1)^{j-l} x_l \frac{n!}{l!(j-l)!(n-j)!}, \qquad (5)$$

$$\left\{\frac{(j-1)(j-2)}{2}a_j+\frac{(j+1)(j+2)}{2}a_{j+2}\right\}=\sum_{l=0}^{j}(-1)^{j-l}y_l\frac{n!}{l!(j-l)!(n-j)!},$$

$$i\cdot\left\{(j+1)(j-1)a_{j+1}\right\}=\sum_{l=0}^{j}(-1)^{j-l}z_l\frac{n!}{l!(j-l)!(n-j)!};$$

If $j=n-1$

$$i\cdot\left\{\frac{(j-1)(j-2)}{2}a_j\right\}=\sum_{l=0}^{j}(-1)^{j-l}x_l\frac{n!}{l!(j-l)!(n-j)!},$$

$$\left\{\frac{(j-1)(j-2)}{2}a_j\right\}=\sum_{l=0}^{j}(-1)^{j-l}y_l\frac{n!}{l!(j-l)!(n-j)!}, \tag{6}$$

$$i\cdot\left\{(j+1)(j-1)a_{j+1}\right\}=\sum_{l=0}^{j}(-1)^{j-l}z_l\frac{n!}{l!(j-l)!(n-j)!};$$

If $j=n$

$$i\cdot\left\{\frac{(j-1)(j-2)}{2}a_j\right\}=\sum_{l=0}^{j}(-1)^{j-l}x_l\frac{n!}{l!(j-l)!(n-j)!},$$

$$\left\{\frac{(j-1)(j-2)}{2}a_j\right\}=\sum_{l=0}^{j}(-1)^{j-l}y_l\frac{n!}{l!(j-l)!(n-j)!},\ 0=\sum_{l=0}^{j}(-1)^{j-l}z_l\frac{n!}{l!(j-l)!(n-j)!}. \tag{7}$$

Equations (5), (6) and (7) will form a system of equations with a lower triangular matrix, which is solved by sequential substitution of values that were found.

Cubic isotropic Bezier curve will have the next form:

$$\mathbf{r}(t)=\mathbf{r_0}(1-t)^3+3\mathbf{r_1}(1-t)^2t+3\mathbf{r_2}(1-t)t^2+\mathbf{r_3}t^3, \tag{8}$$

where $\mathbf{r_0}[(a_0-a_2)i \quad a_0+a_2 \quad -a_1i]$, $\mathbf{r_1}[(a_0-a_2-a_3)i \quad a_0+a_2+a_3 \quad -a_1i]$, $\mathbf{r_2}[(a_0-a_2-2a_3)i \quad a_0+a_2+2a_3 \quad (a_3-a_1)i]$, $\mathbf{r_3}[(a_0-a_2-2a_3)i \quad a_0+a_2+4a_3 \quad (3a_3-a_1)i]$.

When you set the cubic isotropic Bezier curve on the basis of the Weierstrass method on the basis of analytic functions, 12 coordinates of the characteristic quadrilateral are calculated on the basis of 4 independent complex parameters, making it impossible to have an interactive control of the curve form. For the isotropic Bezier curve of the nth order $3(n+1)$ the points' coordinates of characteristic polygon are set on the basis of $(n+1)$ independent complex values.

- Formation of the isotropic Bezier curve on the basis of the plane real curve

Let's model a spatial isotropic curve using the real plane parametric curve. As initial parameters we will take the plane Bezier curve, i.e. we will set equation of the curve on the plane $x=x(t)$, $y=y(t)$. For the realization condition of isotropy the third coordinate is found on the basis of the expression [5]: $z(t)=i\int\sqrt{(x'(t))^2+(y'(t))^2}dt$. In general case (i.e. when the curve is not a Pythagorean-hodograph curve) it's impossible to get the expression without additional calculations, so we apply approximation methods.

We will solve the problem in two stages: on the first stage we will get the polynomial that approximates the function of the square root, and on the second stage we will get the integral.

One of the major disadvantages when we use approximation methods of constructing of spatial curve is the fact that only two coordinates: $x=x(t)$ and $y=y(t)$ will correspond to Bezier curve, third coordinate $z(t)$ will be in the form of an arbitrary polynomial. It's impossible to manage these curves interactively.

- Formation of an spatial isotropic Bezier curve on the basis of isotropic sides of the characteristic polygon

Let's consider the formation of spatial Bezier curve of the nth order when isotropic characteristics are set. As isotropic characteristics may be characteristic sides and chord of the polygon. Let's find limitations that must be imposed on the coordinates of the reference points of the Bezier curve. To do this, let's take the derivative of equation (2) and substitute to the expression (1):

$$\sum_{r=x,y,z}n^2\Big[\sum_{j=0}^{n-1}(r_{j+1}-r_j)^2J_{n-1,j}^2(t)+2\sum_{j=0}^{n-1}\sum_{l=j+1}^{n-1}(r_{j+1}-r_j)J_{n-1,j}(t)(r_{l+1}-r_l)J_{n-1,l}(t)\Big]=0. \tag{9}$$

Expression (9) – is a condition for spatial isotropic Bezier curve of the nth order.

This condition will not be dependent of the values of the parameter *t* if we equate to zero the coefficients of t^j.

Let's write the system of equations that defines the condition of isotropy for the coefficients of Bezier curve of the nth order. The number of conditions is: $j=0..(2n-1)$.

If $j=2k$, then the coefficient of $t^j(1-t)^{n-1-j}$ is calculated on the basis of correlation:

$$n^2\left[(r_{j+1}-r_j)^2\left(\frac{(n-1)!}{j!(n-1-j)!}\right)^2+2\sum_{l=j+1}^{n-1}(r_{j+1}-r_j)\left(\frac{(n-1)!}{j!(n-1-j)!}\right)(r_{l+1}-r_l)\left(\frac{(n-1)!}{l!(n-1-l)!}\right)\right]=0; \tag{10}$$

If $j=2k-1$, then the coefficient of $t^j(1-t)^{n-1-j}$ is calculated as follows:

$$n^2\left[2\sum_{l=j+1}^{n-1}(r_{j+1}-r_j)\left(\frac{(n-1)!}{j!(n-1-j)!}\right)(r_{l+1}-r_l)\left(\frac{(n-1)!}{l!(n-1-l)!}\right)\right]=0. \quad (11)$$

Equations (10) and (11) constitute a system of equations to determine limitations of the values of the reference points of the spatial isotropic Bezier curve of the n[th] order.

If all sides of the characteristic polygon of the curve are isotropic, i.e. for all $i = 0..(n-1)$ are executed the correlation $\sum_{r=x,y,z}(r_{i+1}-r_i)^2=0$ and condition $\sum_{r=x,y,z}(r_0-r_n)^2=0$ (isotropy of the chord that subtends the first and last point of the polygon), then we will have a system of equations:

$$n^2\left[2\sum_{l=j+1}^{n-1}(r_{j+1}-r_j)\left(\frac{(n-1)!}{j!(n-1-j)!}\right)(r_{l+1}-r_l)\left(\frac{(n-1)!}{l!(n-1-l)!}\right)\right]=0 \quad (12)$$

For cubic spatial isotropic Bezier curve of the 3[d] order we will have:

$$\sum_{r=x,y,z}(r_1-r_0)^2=0,\quad \sum_{r=x,y,z}(r_1-r_0)(r_2-r_1)=0,\quad 2\sum_{r=x,y,z}(r_2-r_1)^2+\sum_{r=x,y,z}(r_1-r_0)(r_3-r_2)=0,$$
$$\sum_{r=x,y,z}(r_2-r_1)(r_3-r_2)=0,\quad \sum_{r=x,y,z}(r_3-r_2)^2=0. \quad (13)$$

Expression (13) determines the conditions of isotropy of the spatial Bezier curve of the 3[d] order.

Analysis of the expression (13) shows that the length of the first and last link of the characteristic quadrilateral must be equal to zero, i.e. it must be isotropic.

If $\sum_{r=x,y,z}(r_2-r_1)^2=0$, then the chord that subtends the curve segment will be isotropic.

- Modeling of spatial isotropic Bezier curve on the basis of deformation of the plane curve

Let's deform the plane isotropic curve so that the length of the curve in the complex space is remained unchanged, i.e. the curve is remained isotropic. For a plane isotropic Bezier curve all sides of the characteristic polygon and chord are equal to zero, and for spatial curve, this condition should not be stored, so let's make the method to have a possibility to change the points of the characteristic polygon to store the conditions of isotropy length of the curve. Let's set the determined coordinates for spatial curve as follows: z_0, $\mathbf{r_j}=\lfloor x_j \quad y_j \rfloor$, where $j=\frac{n+3}{2}$.

Let's consider the first condition of isotropy of Bezier curve of the n[th] order, namely the isotropy of sides of characteristic polygon, which match with the direction of the tangent at the point $\mathbf{r_0}$: $\sum_{r=x,y,z}(r_1-r_0)^2=0$. According to initial input data only coordinate z_1 is undetermined in this expression. Considering that for the plane curve: $(x_1-x_0)^2+(y_1-y_0)^2=0$ then the first condition will be the next: $z_1=z_0$.

Let's consider the second condition: $\sum_{r=x,y,z}(r_1-r_0)(r_2-r_1)=0$. Substituting the first condition in this expression we get the equation that is equal to zero. So the second condition allows setting an arbitrary coordinate z_2, which is defined and depends on the order of the curve. The following undetermined coordinates are defined with sequential substitution of coordinates that in the previous equations were found. So that we get a method that allows you to find the isotropic Bezier curve of the n[th] order without solving the quadratic equation (13). For a cubic spatial Bezier curve the following undetermined coordinates will be getting on the basis of the following correlations:

$$y_3=\frac{-2(z_2-z_1)^2-(x_1-x_0)(x_3-x_2)}{(y_1-y_0)}+y_2;$$
$$z_3=\frac{2(y_2-y_1)(z_2-z_1)}{(y_1-y_0)}+z_2; \quad (14)$$
$$x_3=\frac{-(z_2-z_1)^2-(y_2-y_1)^2}{(x_1-x_0)}+x_2.$$

When we apply the method of Weierstrass, the surface is constructed on the basis of the isotropic guide curve and changing *t=u+iv*. We could call the change - quasiconformal, if the parameter *t* will be substituted instead of the complex variable *ku+iv* and *u+ikv*: *t=ku+iv* or *t=u+ikv*. As the guide curve we will use cubic isotropic Bezier curve. When we change *t=u+ikv* then we get:

$$\begin{aligned}\mathbf{r}_{\mathrm{Re}}(u,v)&=\mathbf{r}_{0\,\mathrm{Re}}(1-3u+3u^2-3v^2k^2-u^3+3uv^2k^2)-\mathbf{r}_{0\,\mathrm{Im}}(-3vk+6uvk-3u^2vk+v^3k^3)-\\&-(-3\mathbf{r}_{1\,\mathrm{Re}}(1-2u+u^2-v^2k^2)+3\mathbf{r}_{1\,\mathrm{Im}}(-2vk+2uvk))u+(-3\mathbf{r}_{1\,\mathrm{Im}}(1-2u+u^2-v^2k^2)-\\&-3\mathbf{r}_{1\,\mathrm{Re}}(-2vk+2uvk))vk-(-3\mathbf{r}_{2\,\mathrm{Re}}(1-u)-3\mathbf{r}_{2\,\mathrm{Im}}vk)(u^2-v^2k^2)+2(-3\mathbf{r}_{2\,\mathrm{Im}}(1-u)+\\&+3\mathbf{r}_{2\,\mathrm{Re}}vk)uvk+\mathbf{r}_{3\,\mathrm{Re}}(u^3-3uv^2k^2)-\mathbf{r}_{3\,\mathrm{Im}}(3u^2vk-v^3k^3).\end{aligned} \quad (15)$$

When we change *t = ku + iv* then we get:

$$\begin{aligned}\mathbf{r}_{\mathrm{Re}}(u,v)&=\mathbf{r}_{0\,\mathrm{Re}}(1-3uk+3u^2k^2-3v^2-u^3k^3+3uv^2k)-\mathbf{r}_{0\,\mathrm{Im}}(-3v+6uvk-3u^2vk^2+v^3)-\\&-(-3\mathbf{r}_{1\,\mathrm{Re}}(1-2uk+u^2k^2-v^2)+3\mathbf{r}_{1\,\mathrm{Im}}(-2v+2uvk))uk+(-3\mathbf{r}_{1\,\mathrm{Im}}(1-2uk+u^2k^2-v^2)-\\&-3\mathbf{r}_{1\,\mathrm{Re}}(-2v+2uvk))v-(-3\mathbf{r}_{2\,\mathrm{Re}}(1-uk)-3\mathbf{r}_{2\,\mathrm{Im}}v)(u^2k^2-v^2)+2(-3\mathbf{r}_{2\,\mathrm{Im}}(1-uk)+\\&+3\mathbf{r}_{2\,\mathrm{Re}}v)uvk+\mathbf{r}_{3\,\mathrm{Re}}(u^3k^3-3uv^2k)-\mathbf{r}_{3\,\mathrm{Im}}(3u^2vk^2-v^3).\end{aligned} \quad (16)$$

Let isotropic Bezier curve is constructed using equation (8) on the basis of analytic functions. Let's substitute the values of reference points (8) in equations (15) and (16). Let's calculate the coefficients of the first and the second quadratic forms.

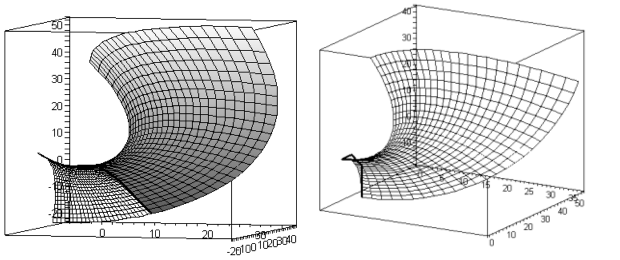
For $t=u+ikv$ coefficients of the first quadratic form will be:

$$E = 9a_{3\mathrm{Re}}^2[2u^2v^2k^2 + v^4k^4 + 2v^2k^2 + 1 + 2u^2 + u^4] + 9a_{3\mathrm{Im}}^2[v^4k^4 + 2v^2k^2 + 2v^2k^2u^2 + 1 + 2u^2 + u^4],$$

$$G = 9a_{3\mathrm{Re}}^2[u^4k^2 + 2u^2v^2k^4 + 2u^2k^2 + v^4k^6 + 2v^2k^4 + k^2] + 9a_{3\mathrm{Im}}^2[2u^2v^2k^4 + v^4k^6 + 2v^2k^4 + u^4k^2 + 2u^2k^2 + k^2],$$

$$F = 0.$$

Analysis of the expressions shows that $E \neq G \Rightarrow k^2E = G$. If you calculate the coefficients of the second quadratic form then we get the following dependencies: $k^2L = -N$, $M = \frac{a_{3\mathrm{Re}}Lk}{a_{3\mathrm{Im}}}$.



a) b)

Fig.1. Minimal surfaces on the basis of isotropic Bezier curve

Mean curvature of the surface in this case is equal to 0, i.e. we will have $H = 0$.

Similar research can be conducted with the change $t=ku+iv$. On the Fig.1a) is shown the two surfaces with different coefficients k for change $t=u+ikv$: $k=-1.5$ and $k=0.5$. On the Fig. 1b) is shown the surface that is classified as the lines of curvature.

Conclusions. Researches have shown that applying the method of Bezier for the construction of the isotropic curve increased the number of points that can be controlled and allows you to have an interactively control of the curve shape. When we use the quasiconformal change of parameter in the equation of the isotropic curve we get the minimal surface with H = 0, but also with $E \neq G$.

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Аушева Н.Н.

Моделирование минимальных поверхностей на основе изотропных кривых и квазиконформной замены параметра

Аннотация. В работе рассматривается построение минимальных поверхностей на основе пространственных изотропных кривых Безье. Моделирование изотропных кривых осуществляется в комплексном пространстве. Предлагается несколько способов построения таких кривых: на основе аналитической функции комплексной переменной, на основе заданной действительной кривой на плоскости, на основе деформации плоской кривой в комплексном пространстве. Найдены условия изотропности для кривой Безье n-го порядка и приведен пример для кривой третьего порядка. Рассмотрен частный случай построения кривой Безье на основе изотропных сторон характеристического четырехугольника, в этом случае действительная кривая будет лежать на плоскости в трехмерном пространстве. Полученные кривые были использованы в качестве направляющей кривой для построения поверхностей. Для построения поверхности производится квазиконформная замена параметра в уравнении изотропной кривой. Путем выделения действительной и мнимой части получаем уравнения поверхностей. Проведен анализ коэффициентов первой и второй квадратичных форм. Было доказано, что полученные поверхности являются минимальными поверхностями.

Ключевые слова: *изотропные кривые, кривые Безье, минимальные поверхности*